APPLICATION OF THE INTERVAL METHODS FOR SOLVING LINEAR THERMAL DIFFUSION PROBLEMS

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Abstract. The example of two-dimensional non-steady state heat flow using the interval arithmetic and the 1st scheme of the boundary element method is presented. This example is a typical linear task, where the boundary of the homogeneous domain is an interval. In the final part of the paper, results of numerical computations are shown with the different boundary conditions.

1. Governing equation

Let us consider a two-dimensional homogeneous domain. The boundary of the domain considered can be described in an approximated way using an interval, namely $\langle \Gamma^-, \Gamma^+ \rangle$, where $\Gamma^- = \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3^-$ and $\Gamma^+ = \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_3^+$ denote the first and the second endpoints of the interval, respectively - see Figure 1 [1].

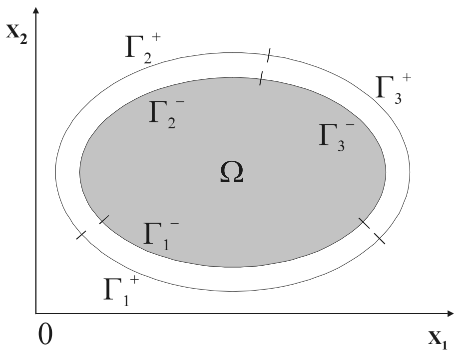

Fig. 1. Domain considered

The temperature field in 2D domain oriented in Cartesian co-ordinate system is described by the following linear energy equations

$$x \in \Omega: \quad c\rho \frac{\partial T(x,\ t)}{\partial t} = \lambda \nabla^2 T(x,\ t) \tag{1}$$

where c is the specific heat, ρ is the mass density, λ is the thermal conductivity, T is the temperature, $x=\{x_1, x_2\}$ is the spatial co-ordinate and t is the time.
The above equation is supplemented by the following boundary-initial conditions:

$$
\begin{aligned}
&x \in \langle \Gamma_1^-, \Gamma_1^+ \rangle: \quad T(x,\ t) = T_b \\
&x \in \langle \Gamma_2^-, \Gamma_2^+ \rangle: \quad q(x,\ t) = -\lambda \frac{\partial T(x,\ t)}{\partial n} = q_b \\
&x \in \langle \Gamma_3^-, \Gamma_3^+ \rangle: \quad q(x,\ t) = -\lambda \frac{\partial T(x,\ t)}{\partial n} = \alpha \left[T(x,\ t) - T^\infty \right] \\
&t = 0: \quad T(x,\ t) = T_0(x)
\end{aligned}
\tag{2}
$$

where T_b is the known boundary temperature, $\partial T(x,\ t)/\partial n$ is the normal derivative at the boundary point x, q_b is the given boundary heat flux, α is the heat transfer coefficient, T^∞ is the ambient temperature, T_0 is the initial temperature.

2. Interval boundary element method

In this paper the 1st scheme of the boundary element method is used. As first, the time grid must be introduced

$$
0 = t^0 < t^1 < t^2 < \ldots < t^{f-1} < t^f < \ldots < t^F < \infty \tag{3}
$$

with a constant time step $\Delta t = t^f - t^{f-1}$.
The basic idea of the 1st scheme of the boundary element method consists in 'step by step' integration with respect to time.
For the problem analysed the boundary integral equation for transition $t^{f-1} \to t^f$ corresponding to the equation (1) is of the following form [2-4]

$$
\begin{gathered}
B(\xi) T(\xi,\ t^f) + \frac{1}{c\rho} \int_{t^{f-1}}^{t^f} \int_{\langle \Gamma^-, \Gamma^+ \rangle} T^*(\xi,\ x,\ t^f,\ t) q(x,\ t) \mathrm{d}\Gamma \mathrm{d}t = \\
\frac{1}{c\rho} \int_{t^{f-1}}^{t^f} \int_{\langle \Gamma^-, \Gamma^+ \rangle} q^*(\xi,\ x,\ t^f,\ t) T(x,\ t) \mathrm{d}\Gamma \mathrm{d}t + \iint_{\langle \Omega^-, \Omega^+ \rangle} T^*(\xi,\ x,\ t^f,\ t) T(x,\ t^{f-1}) \mathrm{d}\Omega
\end{gathered}
\tag{4}
$$

where ξ is the observation point, $B(\xi) \in (0,\ 1]$, x is the point under consideration, $q = -\lambda \partial T / \partial n$ is the boundary flux, T^* is a fundamental solution of the form

$$
T^*(\xi,\ x,\ t^f,\ t) = \frac{1}{4\pi a(t^f - t)} \exp\left[-\frac{r^2}{4a(t^f - t)} \right] \tag{5}
$$

where $a = \lambda / c\rho$ is the diffusion coefficient and r is the distance between the points ξ and x

$$r = \sqrt{\left(x_1 - \xi_1\right)^2 + \left(x_2 - \xi_2\right)^2} \tag{6}$$

The normal heat flux resulting from the fundamental solution should be found in analytic way and then

$$q^*(\xi,\ x,\ t^f,\ t) = -\lambda \frac{\partial T^*(\xi,\ x,\ t^f,\ t)}{\partial n} = \frac{\lambda}{8\pi a^2 \left(t^f - t\right)^2} \exp\left(-\frac{r^2}{4a(t^f - t)}\right) \tag{7}$$

where $d = \left(x_1 - \xi_1\right)\cos\alpha_1 + \left(x_2 - \xi_2\right)\cos\alpha_2$ and $\cos\alpha_1$, $\cos\alpha_2$ are the directional cosines of the normal outward vector n.
The numerical approximation of this equation leads to the following system of interval equations

$$\mathbf{G} \cdot \mathbf{q}^f = \mathbf{H} \cdot \mathbf{T}^f + \mathbf{P} \cdot \mathbf{T}^{f\text{-}1} \tag{8}$$

where $\mathbf{q}^f = \left\langle \mathbf{q}^-, \mathbf{q}^+ \right\rangle^f$ and $\mathbf{T}^f = \left\langle \mathbf{T}^-, \mathbf{T}^+ \right\rangle^f$ are the vectors of the boundary values (heat fluxes and temperatures), $\mathbf{G} = \left\langle \mathbf{G}^-, \mathbf{G}^+ \right\rangle$, $\mathbf{H} = \left\langle \mathbf{H}^-, \mathbf{H}^+ \right\rangle$, $\mathbf{P} = \left\langle \mathbf{P}^-, \mathbf{P}^+ \right\rangle$ are the matrixes resulting from the numerical approximation of the adequate integrals. The first endpoint of each interval in the system of equations (8) must be always less than or equal to the second endpoint of the interval considered.
The equation (8) can be written as follows

$$\sum_{j=1}^{N} G_{ij}\, q_j^f = \sum_{j=1}^{N} H_{ij}\, T_j^f + \sum_{l=1}^{L} P_{il}\, T_l^f \tag{9}$$

where N is the number of the boundary elements of the boundary $\left\langle \Gamma^-, \Gamma^+ \right\rangle$ and L is the number of the internal cells of the domain considered.
The elements of the matrix **H** takes the form

$$H_{ij} = \begin{cases} \hat{H}_{ij}, & i \neq j \\ \hat{H}_{ij} - 0.5, & i = j \end{cases} \tag{10}$$

The elements of the matrixes **G** and $\hat{\mathbf{H}}$ are described as follows:

$$
\begin{aligned}
G_{ij}^{-} &= \min\left\{\frac{1}{4\pi\lambda}\int\limits_{\langle\Gamma_j^-,\,\Gamma_j^+\rangle}\mathrm{Ei}\left(-\frac{\left(r_{ij}^{-}\right)^2}{4a\Delta t}\right)\mathrm{d}\Gamma_j,\;\; \frac{1}{4\pi\lambda}\int\limits_{\langle\Gamma_j^-,\,\Gamma_j^+\rangle}\mathrm{Ei}\left(-\frac{\left(r_{ij}^{-}\right)^2}{4a\Delta t}\right)\mathrm{d}\Gamma_j\right\}\\
G_{ij}^{+} &= \max\left\{\frac{1}{4\pi\lambda}\int\limits_{\langle\Gamma_j^-,\,\Gamma_j^+\rangle}\mathrm{Ei}\left(-\frac{\left(r_{ij}^{-}\right)^2}{4a\Delta t}\right)\mathrm{d}\Gamma_j,\;\; \frac{1}{4\pi\lambda}\int\limits_{\langle\Gamma_j^-,\,\Gamma_j^+\rangle}\mathrm{Ei}\left(-\frac{\left(r_{ij}^{-}\right)^2}{4a\Delta t}\right)\mathrm{d}\Gamma_j\right\}
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\hat{H}_{ij}^{-} &= \min\left\{\begin{array}{l}
\dfrac{1}{2\pi}\displaystyle\int\limits_{\langle\Gamma_j^-,\,\Gamma_j^+\rangle}\dfrac{d_{ij}^{-}}{\left(r_{ij}^{-}\right)^2}\exp\left(-\dfrac{\left(r_{ij}^{-}\right)^2}{4a\Delta t}\right)\mathrm{d}\Gamma_j,\\
\dfrac{1}{2\pi}\displaystyle\int\limits_{\langle\Gamma_j^-,\,\Gamma_j^+\rangle}\dfrac{d_{ij}^{+}}{\left(r_{ij}^{+}\right)^2}\exp\left(-\dfrac{\left(r_{ij}^{+}\right)^2}{4a\Delta t}\right)\mathrm{d}\Gamma_j
\end{array}\right\}\\
\hat{H}_{ij}^{+} &= \max\left\{\begin{array}{l}
\dfrac{1}{2\pi}\displaystyle\int\limits_{\langle\Gamma_j^-,\,\Gamma_j^+\rangle}\dfrac{d_{ij}^{-}}{\left(r_{ij}^{-}\right)^2}\exp\left(-\dfrac{\left(r_{ij}^{-}\right)^2}{4a\Delta t}\right)\mathrm{d}\Gamma_j,\\
\dfrac{1}{2\pi}\displaystyle\int\limits_{\langle\Gamma_j^-,\,\Gamma_j^+\rangle}\dfrac{d_{ij}^{+}}{\left(r_{ij}^{+}\right)^2}\exp\left(-\dfrac{\left(r_{ij}^{+}\right)^2}{4a\Delta t}\right)\mathrm{d}\Gamma_j
\end{array}\right\}
\end{aligned}
\tag{12}
$$

The vector **P** connected with the distribution of searched function in the interior of the domain considered at the moment t^{f-1} is described by the integrals

$$
\begin{aligned}
P_{il}^{-} &= \min\left\{\begin{array}{l}
\dfrac{1}{4\pi a\Delta t}\displaystyle\iint\limits_{\langle\Omega_l^-,\,\Omega_l^+\rangle}\exp\left(-\dfrac{\left(r_{il}^{-}\right)^2}{4a\Delta t}\right)\mathrm{d}\Omega_l,\\
\dfrac{1}{4\pi a\Delta t}\displaystyle\iint\limits_{\langle\Omega_l^-,\,\Omega_l^+\rangle}\exp\left(-\dfrac{\left(r_{il}^{+}\right)^2}{4a\Delta t}\right)\mathrm{d}\Omega_l
\end{array}\right\}\\
P_{il}^{+} &= \max\left\{\begin{array}{l}
\dfrac{1}{4\pi a\Delta t}\displaystyle\iint\limits_{\langle\Omega_l^-,\,\Omega_l^+\rangle}\exp\left(-\dfrac{\left(r_{il}^{-}\right)^2}{4a\Delta t}\right)\mathrm{d}\Omega_l,\\
\dfrac{1}{4\pi a\Delta t}\displaystyle\iint\limits_{\langle\Omega_l^-,\,\Omega_l^+\rangle}\exp\left(-\dfrac{\left(r_{il}^{+}\right)^2}{4a\Delta t}\right)\mathrm{d}\Omega_l
\end{array}\right\}
\end{aligned}
\tag{13}
$$

The interval Gauss elimination method with the decomposition procedure [1] has been used to solve the system of equations (8).

3. Interval Gauss elimination method

Let us consider the system of equations

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{B} \tag{14}$$

where

$$\mathbf{A} = \sum_{j=1}^{N} \left\langle G_{ij}^{-},\, G_{ij}^{+} \right\rangle \tag{15}$$

and

$$\mathbf{B} = \sum_{j=1}^{N} \left\langle H_{ij}^{-},\, H_{ij}^{+} \right\rangle \left\langle T_{j}^{-},\, T_{j}^{+} \right\rangle^{f} + \sum_{l=1}^{L} \left\langle P_{il}^{-},\, P_{il}^{+} \right\rangle \left\langle T_{l}^{-},\, T_{l}^{+} \right\rangle^{f-1} \tag{16}$$

We use a decomposition to solve the linear system of equations (14) [1]. The main matrix **A** must be written as a product of two matrices **L** and **U**, where **L** is lower triangular and **U** is upper triangular, this means

$$L_{i,j} = \begin{cases} \left\langle L_{ij}^{-},\, L_{ij}^{+} \right\rangle, & i > j \\ 1, & i = j \\ 0, & i < j \end{cases} \tag{17}$$

and

$$U_{ij} = \begin{cases} \left\langle U_{ij}^{-},\, U_{ij}^{+} \right\rangle, & i \le j \\ 0, & i > j \end{cases} \tag{18}$$

where $i, j = 0, 1, \ldots, n$ (n is the dimension of the matrix).
The system of equations (14) takes a form

$$\mathbf{A} \cdot \mathbf{x} = (\mathbf{L} \cdot \mathbf{U}) \cdot \mathbf{x} = \mathbf{L} \cdot (\mathbf{U} \cdot \mathbf{x}) = \mathbf{B} \tag{19}$$

After determining the 'missing' boundary values the functions $T(x, t)$ at internal nodes of the domain considered are calculated separately. At each node of the domain the temperature is given as an interval

$$T(\xi,\, t^{f}) = \left\langle \min\left\{ T(\xi^{-},\, t^{f}),\, T(\xi^{+},\, t^{f}) \right\},\, \max\left\{ T(\xi^{-},\, t^{f}),\, T(\xi^{+},\, t^{f}) \right\} \right\rangle \tag{20}$$

4. Results of computations

Let us consider a 2D domain of dimension $d_1 = 0.1$ m and $d_2 = 0.1$ m. The right boundary of this square is an interval, where $L^- = L - 0.01 \cdot L$, $L^+ = L + 0.01 \cdot L$. On this side of the domain considered the boundary condition of the first or second type is assumed: $T_b = 500$°C, $q_b = 10000$ W/mK.
On the rest of boundaries the boundary condition of the first type is assumed: $T_b = 500$°C. The following input data have been introduced: $\lambda = 330$ W/mK, $c = 420$ J/kg · K, $\rho = 8920$ kg/m^3, initial temperature $T_0 = 1000$°C. Each side of the domain considered has been divided into 8 linear elements, the interior of this domain has been divided into 64 elements, the time step $\Delta t = 2$ s.
Figures 2 and 3 illustrate the cooling curves obtained at the point $x_1 = 0.375$ m, $x_2 = 0.625$ m in the domain considered for the boundary condition of the first and second type respectively (Tem_L, Tem_R denote the first and the second end-points of the interval).

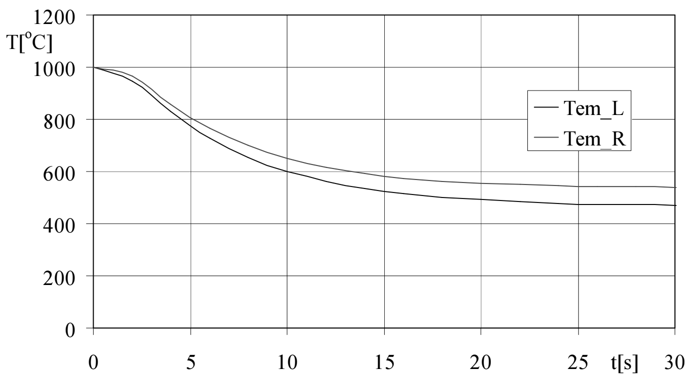

Fig. 2. Cooling curves for the first example

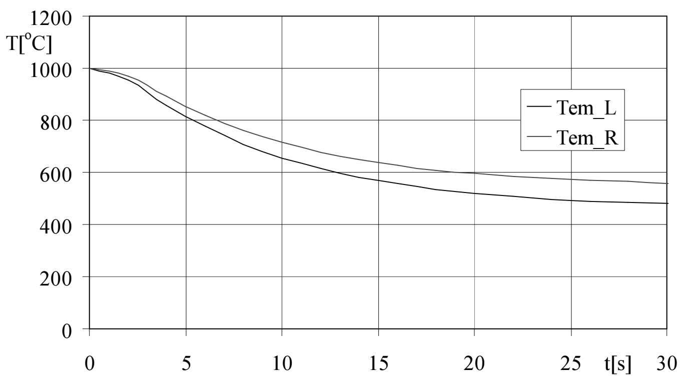

Fig. 3. Cooling curves for the second example

Figures 4 and 5 illustrate the courses of cooling curves obtained at the same point x_1, x_2 using the interval method for different boundary conditions (Tem is the arithmetic average of the values Tem_L and Tem_R) and obtained by classical way (Tem_n).

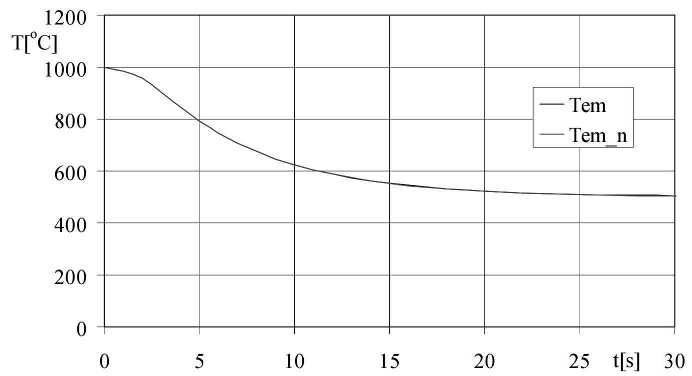

Fig. 4. Cooling curves for the first example

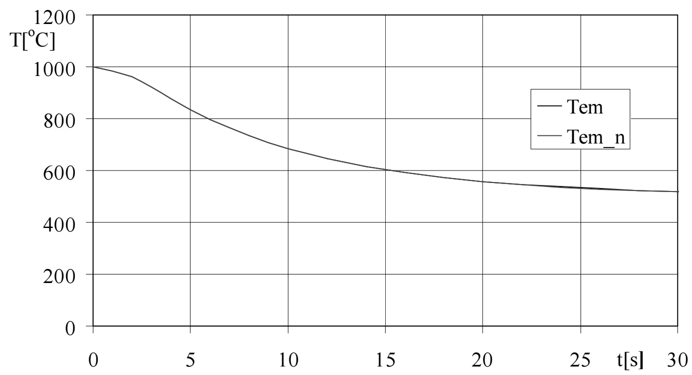

Fig. 5. Cooling curves for the second example

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