Please cite this article as:
Ewa Węgrzyn-Skrzypczak, Tomasz Skrzypczak, Accuracy of numerical solution of heat diffusion equation, Scientific Research of the Institute of Mathematics and Computer Science, 2008, Volume 7, Issue 1, pages 193-198.
The website: http://www.amcm.pcz.pl/

ACCURACY OF NUMERICAL SOLUTION OF HEAT DIFFUSION EQUATION

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Abstract. Presented paper is focused on results of numerical solution of transient heat conductivity equation in two-dimensional region. Convection term is neglected in mathematical model of the phenomenon. Solutions based on classical Galerkin finite element formulation obtained for girds of different qualities are compared to discontinuous Galerkin method. Spatial discretization of computational domain and order of basis functions are taken into account.

Introduction

High accuracy of numerical solutions of differential equations is one of the priority in modern calculation techniques commonly using in engineering practice. From among of many advanced numerical methods applied to calculation most often finite difference method FDM [1, 2], finite element method FEM [1, 3, 4], boundary element method BEM [5], finite volume method FVM [6] are used. In the last decade the huge growth interested of discontinuous Galerkin method DGM, which can be localized among FEM and FVM, is observed. In 1973, Reed and Hill [7] and also independently Lesaint and Raviart [8] described DGM. The name of the method concerns to locality of used solution, which can take into consideration discontinuity in scale of particular element. FEM allows to obtain only global solution for all nodes in the grid, while DGM permits to get local solution for single element. Appropriate introduction of boundary condition on the edges of finite elements leads to continuous as well as discontinuous solution. This feature of the method enables to use meshes containing "hanging nodes" in calculation process. Mesh refinement (h-adaptation) and mesh enrichment (p-adaptation) are also easier. In presented paper results for FEM and DGM implemented for transient heat transfer problem with Newton boundary condition in square testing domain are presented and compared.

1. Formulation of the problem

Partial differential equation of heat diffusion in two-dimensional domain is considered and written in following form

$$\frac{\partial}{\partial x}\cdot\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\cdot\left(\lambda\frac{\partial T}{\partial y}\right)=c\rho\frac{\partial T}{\partial t} \tag{1}$$

where T[K] denotes temperature, t [s] - time, λ [W/mK] - thermal conductivity, c[J/kgK] - specific heat, ρ[kg/m^3] - density.

Equation (1) are completed by appropriate boundary condition:

$$-\lambda\left.\frac{\partial T}{\partial n}\right|_{\Gamma}=\alpha\left(T-T_{\infty}\right) \tag{2}$$

where n is a direction of vector normal to the external boundary, α [W/m^{2}K] denotes heat transfer coefficient and T_{∞}[K] is ambient temperature.

Initial condition are defined as follows:

$$\left.T\right|_{t=0}=T_0 \tag{3}$$

where T_0 [K] is initial temperature.

In discontinuous Galerkin method procedure leading to spatial discretization of differential equation is similar to analogous procedure employed for finite element method. Equation (1) after multiply by weighting function w and integrate over Ω domain is written in weak form:

$$\lambda\int_{\Omega}\left(\frac{\partial w}{\partial x}\cdot\frac{\partial T}{\partial x}\right)d\Omega+\lambda\int_{\Omega}\left(\frac{\partial w}{\partial y}\cdot\frac{\partial T}{\partial y}\right)d\Omega-\int_{\Gamma}wF\left(T\right)\cdot\mathbf{n}d\Gamma-c\rho\int_{\Omega}w\frac{\partial T}{\partial t}=0 \tag{4}$$

Domain Ω after spatial discretization procedure is divided into N elements:

$$\Omega=\bigcup_{j=1}^{N}\Omega_j \tag{5}$$

Solution T is approximated in interior of element using polynomial functions of p degree. Basis functions are orthogonal in $L^2(\Omega_j)$ space [9]. Because of discontinuity in solution appearing on the edges of element numerical flux must be introduced. Value of the flux depends on solution $T(\Omega_j)$, $T(\Omega_k)$ obtained for elements Ω_j, Ω_k which share Γ_{jk} edge and it may be written in following form

$$F_n\left(T\left(\Omega_j\right),T\left(\Omega_k\right)\right)=F\left(T\right)\cdot\mathbf{n} \tag{6}$$

Integral terms occurring in equation (4) may be calculated using one of the commonly used method of numerical integration [3, 4].

2. Examples of calculation

Testing square shaped domain 0.2x0.2 [m], introduced initial, boundary conditions and material properties are presented in Figure 1.

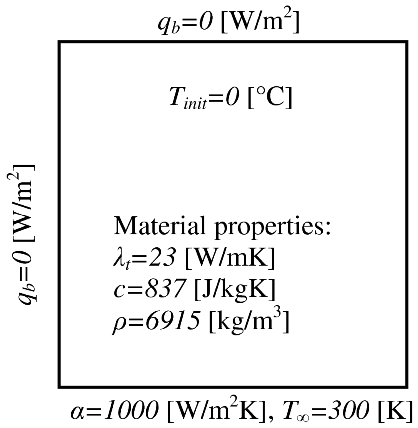

Fig. 1. Geometry of the testing domain

Calculations were performed on structural grids with different quality of spatial discretization. DG method was tested on grids presented in Figure 2, on gird 10x10 calculations were realized for p = 2,3,4, on grid 20x20 for p = 2,3, on grid 40x40 for p = 2, where p means order of approximation. FEM was tested additionally for grids 80x80 and 160x160, the last solution was treated as a exact solution. Calculations in all cases were performed for 100 s with constant time step Δt = 0.01 s.

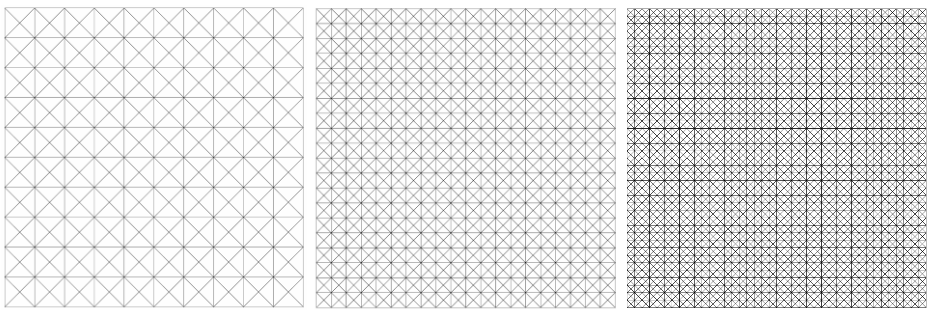

Fig. 2. Spatial discretization of the testing domain 10x10, 20x20, 40x40

In DG method accuracy of temperature approximation increases along with polynomial's interpolation order as well as number of interpolation nodes in the interior of finite element (Fig. 3). Relation between interpolation order p and node number n describes following formula

$$n = 0.5(p+1)(p+2) \tag{7}$$

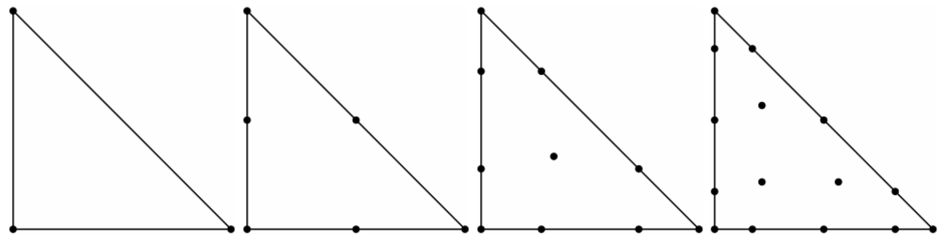

Fig. 3. Distribution of the interpolation nodes in triangular element for p = 1..4

Changes of temperature in function of time on the heated wall are presented in Figure 4. Accurate results obtained with FEM were gained on 40x40 mesh. DGM allows to obtain accurate results on 10x10 mesh with polynomial order $p > 2$.

Temperature values on heated boundary for t = 10, 25, 50, 100 s are compiled in Table 1. In the last column the most accurate results are presented.

Table 1.

Comparison of temperature results obtained for heated boundary

T, s	10x10				20x20			40x40		80x 80	160x 160
	p = 1	p = 2	p = 3	p = 4	p = 1	p = 2	p = 3	p = 1	p = 2	p = 1	p = 1
10	54.6	78.4	74.5	74.0	69.5	74.6	74.2	73.4	74.2	74.0	74.2
25	95.4	105.6	104.6	104.5	104.1	104.6	104.6	104.3	104.5	104.5	104.5
50	128.9	131.4	131.4	131.4	131.1	131.3	131.4	131.3	131.4	131.3	131.4
100	160.0	159.8	160.0	160.0	159.9	159.9	160.0	159.9	160.0	159.9	159.9

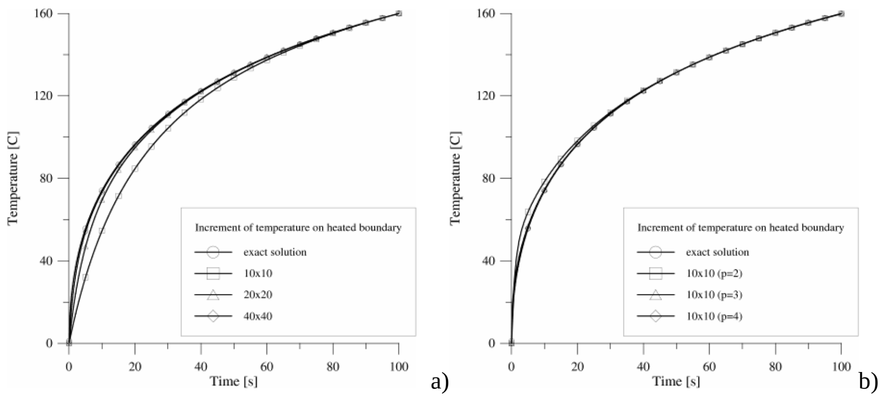

Fig. 4. Comparison of exact solution to FEM solutions obtained for meshes of lower qualities (a) and to DGM solutions for coarse 10x10 mesh and p = 2..4 (b)

Conclusions

Presented results shows that discontinuous Galerkin method applied to higher order approximation allows to obtain accurate results even on coarse grids. Appropriate combination of spatial adaptation of the mesh with increasing of polynomial order of basis functions leads to accurate solutions with efficient memory usage and computational cost.

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