ON SOME MATHEMATICAL PROBLEMS OF CLAIMS PROCESSING IN INSURANCE COMPANIES

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Abstract. In the present paper an analysis of models for eqitype and multi-type claims processing, when general number of insurance contracts is function of time, is carried out. The closed queueing networks are the models for claims processing. The problem of optimal number of estimators on definite intervals of time is considered.

1. Introduction

In the present paper we consider the mathematical models for eqitype and multi-type claims processing. At the end of the paper we raise some problems, which we are going to solve in the nearest time. We consider the work of an insurance company on the interval of time $[0,T]$. Assume that the number of contracts concluded with clients at the moment of time t, $t \le T$, is some definite function of time $K(t)$. Notice, that for the first time the detailed problem statement and its solution when the number of contracts is constant, i.e. when $K(t) = K$, was given in [1]. Therefore we omit some details in the present paper. However, it should be noted, that the case when $K(t)$ is functions of time corresponds to the real conditions of an insurance company's functioning more, so the considering in the paper model is more adequate. Note, that research on multi-type claims processing model in the case $K(t) = K$ was done in [2, 3].

2. Models for eqitype clams

In the case of insurance incident an insurer raises a claim. This claim may be in two stages: the stage of estimation and the stage of payment. Let the number of company employees who estimate claims be $n-1$. We call them estimators. Insurer's claims may be in one of the following states: C_0 - a claim is not raised, C_1 - a claim is in the stage of estimation, C_2 - a claim is in the stage of payment. We need the random process $k(t)$ to be a Markov process, so assume that the transition probability from the state C_0 into the state C_1 on the interval of time $[t, t+\Delta t]$ is $\mu_{01}(t)\Delta t + o(\Delta t)$, where $\mu_{01}(t)$ is intensity of a such transition and $\mu_{01}(t)$ is piecewise analytical function of time with two intervals of constancy:

$$\mu_{01}(t) = \begin{cases} \mu_{01}, t \in [0, T/2] \\ \mu_{01}^*, t \in (T/2, T] \end{cases}$$

This follows from the real conditions. For example, it may be one in wintertime and other in summertime. The times of claims processing by estimators and the times of transitions from the state C_2 into the state C_0 are distributed according to exponential rule with intensities μ_1 and μ_2 accordingly. Assume, that at some moment of time *t* our system is in the state $k(t) = (k_1(t), k_2(t))$, if at this moment of time $k_1(t)$ claims are in the state C_1, and $k_2(t)$ claims are in the state C_2.

Let us introduce the following coefficients: D_0 is company's profit per time unit form one insurer, when he does not raise a claim, i.e. the claim is in the state C_0; D_i is company's loss from a claim when it is in the state C_i, $i = 1, 2$; E_1 is a salary of a estimator per time unit; E_2 is a salary of a claim payment cashier per time unit. Then at the moment of time *t* the company's profit is equal to

$$\Pi(t) = D_0(K(t) - \sum_{i=1}^{2} k_i(t)) - \sum_{i=1}^{2} D_i k_i(t) - E_1(n-1) - E_2$$

Since the vector $k(t)$ forms a two-dimensional Markov random process, then $\Pi(t)$ is also a random process. We find an expression for the average income from an insurer on the interval of time $[0, T]$

$$R(t) = \frac{1}{T}\int_0^T M\left\{\frac{\Pi(t)}{K(t)}\right\} dt = D_0 - \frac{1}{T}\int_0^T \left[\sum_{i=1}^{2}(d_i n_i(t) + E_i l_i(t))\right] dt$$

where $d_i = D_0 + D_i$, $n_i(t) = M\left\{\frac{k_i(t)}{K(t)}\right\}$, $i = 1,2$, $l_1(t) = \frac{n-1}{K(t)}$, $l_2(t) = \frac{1}{K(t)}$ and company's average loss is characterized by a functional

$$W(T) = W(T, n) = \frac{1}{T}\int_0^T \left[K(t)\sum_{i=1}^{2}(d_i n_i(t) + E_i l_i(t))\right] dt \tag{1}$$

We are interested in the following problem: to find a number of estimators who have to work during the intervals of time $[0, T/2]$, $(T/2, T]$, so that the average number of claims which are in the states C_1 and C_2 does not exceed $n-1$ and 1 accordingly and loss (1) is minimal:

$$\begin{cases} W(T) \to \min\limits_{n} \\ K(t)n_1(t) \le n-1, t \in [0,T] \\ K(t)n_2(t) \le 1, t \in [0,T] \end{cases} \tag{2}$$

Note that in this case the probabilistic model of the claims processing maybe a closed queue system which consists of three systems with according number of service lines in systems and according transition probabilities between systems. The system S_1 with service lines number $m_1 = n-1$ corresponds to claims processing in the stage of estimation, the system S_2 with service lines number $m_2 = 1$ corresponds to claims payment, the system S_0 corresponds to claims stay in the state C_0; transition probabilities equal to $p_{01} = p_{12} = p_{20} = 1$ accordingly, $p_{ij} = 0$ in other cases. Note that the number of claims in the network does not depend on time.

Assume that $K(t) \ge K^*$, $t \in [0,T]$ and $K^* \ge 1$. It also corresponds to real practice conditions. Using technique stated in [2], it is possible to show that density of probabilities distribution $p(x,t) = p(x_1(t), x_2(t))$ of the vector $\frac{k(t)}{K(t)} = \left(\frac{k_1(t)}{K(t)}, \frac{k_2(t)}{K(t)}\right)$ to an accuracy $\mathrm{O}(\varepsilon^2(t))$, where $\varepsilon(t) = l_2(t)$, satisfies the partial derivatives equation

$$\frac{\partial p(x,t)}{\partial t} = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\big(A_i(x,t)p(x,t)\big) + \frac{\varepsilon(t)}{2}\sum_{i,j=1}^{2}\frac{\partial^2}{\partial x_i \partial x_j}\big(B_{ij}(x,t)p(x,t)\big) + \\ + 2\varepsilon(t)K'(t)p(x,t) \tag{3}$$

where

$$A_i(x,t) = \sum_{j=1}^{2} \mu_j q_{ji} \min(l_j(t), x_j(t)) + \mu_{0i}(t)(1 - \sum_{j=1}^{2} x_j(t)); \mu_{02}(t) = 0, \; q_{ji} = \begin{cases} -1, i = j \\ p_{ij}, i \ne j \end{cases}$$

$$B_{ij}(x,t) = \begin{cases} \sum\limits_{j=1}^{2} \mu_j \min(l_j(t), x_j(t)) + \mu_{0i}(t)(1 - \sum\limits_{j=1}^{2} x_j(t)), i = j \\ -2\mu_i \min(l_i(t), x_i(t)), i = 1, j = 2 \\ 0, \; i = 2, \; j = 1 \end{cases}$$

Note, that when $K(t) = K$, this equation turns into corresponding Fokker-Planc--Kolmogorov equation for this case.

Further we should find a solution of the optimization problem under constraints $A=\{0\le n_1(t)\le l_1(t), 0\le n_2(t)\le l_2(t)\}$. Besides the equality holds:

$$\frac{\partial \min(u,v)}{\partial v}=c(u-v)=\begin{cases}1, u\ge v\\ 0, u<v\end{cases}$$

Therefore it follows from the type of coefficients $B_{ij}(x,t)$ and definitions $l_i(t)$, $x_i(t)$, $i=1,2$, $\mu_{02}(t)$, and bounded density $p(x,t)$ that the part of the expression $\frac{\varepsilon(t)}{2}\sum_{i,j=1}^{2}\frac{\partial^2}{\partial x_i\partial x_j}\left(B_{ij}(x,t)p(x,t)\right)$ concerning the items of the type $\frac{\varepsilon(t)}{2}\frac{\partial^2}{\partial x_i\partial x_j}\left(\mu_i\min(l_i(t),x_i(t))\right)$ in the equation (3) vanishes, so in the region A the equation (3) may be written in the form:

$$\frac{\partial p(x,t)}{\partial t}=-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(A_i(x,t)p(x,t)\right)+\frac{\varepsilon(t)}{2}\mu_{01}(t)\frac{\partial^2}{\partial x_1^2}[(1-x_1(t)-x_2(t))p(x,t)]+$$
$$+2\varepsilon(t)K'(t)p(x,t)$$

Making use of several obvious transformations the last equation may be rewritten in the following way:

$$\frac{\partial p(x,t)}{\partial t}=[\mu_1+\mu_2+\mu_{01}(t)+2\varepsilon(t)K'(t)]p(x,t)+$$
$$+[\mu_1x_1(t)-\mu_{01}(t)(1-x_1(t)-x_2(t))-\varepsilon(t)\mu_{01}(t)]\frac{\partial p(x,t)}{\partial x_1}+ \tag{4}$$
$$+[\mu_2x_2(t)-\mu_1x_1(t))]\frac{\partial p(x,t)}{\partial x_2}+\frac{\varepsilon(t)}{2}\mu_{02}(t)(1-x_1(t)-x_2(t))\frac{\partial^2 p(x,t)}{\partial x_1^2}$$

The equation (4) differs from the Fokker-Planc-Kolmogorov equation only by the expression $2\varepsilon(t)K'(t)$, therefore we can try to find its solution as the density of probabilities distribution of two-dimensional random quantity (it is important to note that we are interested in components of the vector $n(t)$ more than in the solution of the equation (4))

$$p(x,t)=\frac{1}{2\pi}\sqrt{|D(t)|}\exp\left[-\frac{1}{2}\sum_{i,j=1}^{2}(x_i(t)-n_i(t))d_{ij}(t)(x_j(t)-n_j(t))\right]$$

Let us expand the coefficients of the equation (4) in Taylor series in the neighborhood of the point $(n_1(t),n_2(t))$ and not go beyond the first expansion terms:

$$a_1(n,t) = \mu_1 + \mu_2 + \mu_{01}(t) + 2\varepsilon(t)K'(t)$$

$$a_2(n,t) = (\mu_1 + \mu_{01}(t))n_1(t) + \mu_{01}(t)n_2(t) - \mu_{01}(t)(\varepsilon(t)+1)$$

$$a_3(n,t) = \mu_2 n_2(t) - \mu_1 n_1(t)$$

$$a_4(n,t) = \frac{\varepsilon(t)}{2}\mu_{01}(t)(1 - n_1(t) - n_2(t))$$

Let us find the derivatives of the function

$$p(x,t) = p(x_1(t), x_2(t)) = \frac{1}{2\pi}\sqrt{\det D(t)}\,\exp[-\frac{1}{2}[(x_1(t)-n_1(t))^2 d_{11}(t) +$$
$$+ 2(x_1(t)-n_1(t))(x_2(t)-n_2(t))d_{12}(t) + (x_2(t)-n_2(t))^2 d_{22}(t)]] = \alpha(t)e^{-\beta(t)}$$

where $\det D(t) = d_{11}(t)d_{22}(t) - d_{12}^2(t)$:

$$\frac{\partial p(x,t)}{\partial t} = \alpha'(t)e^{-\beta(t)} - \frac{1}{2}\alpha(t)e^{-\beta(t)}[2(x_1(t)-n_1(t))(x_1'(t)-n_2'(t))d_{11}(t) +$$
$$+ (x_1(t)-n_1(t))^2 d_{11}'(t) + 2(x_1'(t)-n_1'(t))(x_2(t)-n_2(t)) + 2(x_1(t)-n_1(t))\times$$
$$\times(x_2'(t)-n_2'(t))d_{12}(t) + 2(x_1(t)-n_1(t))(x_2(t)-n_2(t))d_{12}'(t) +$$
$$+ 2(x_2(t)-n_2(t))(x_2'(t)-n_2'(t))d_{22}(t) + (x_2(t)-n_2(t))^2 d_{22}'(t)$$

$$\frac{\partial p(x,t)}{\partial x_1} = -\alpha(t)e^{-\beta(t)}[(x_1(t)-n_1(t)d_{11}(t) + (x_2(t)-n_2(t)d_{12}(t)]$$

$$\frac{\partial p(x,t)}{\partial x_2} = -\alpha(t)e^{-\beta(t)}[(x_1(t)-n_1(t)d_{12}(t) + (x_2(t)-n_2(t)d_{22}(t)]$$

$$\frac{\partial^2 p(x,t)}{\partial x_1^2} = -\alpha(t)e^{-\beta(t)}[(x_1(t)-n_1(t)d_{11}(t) + (x_2(t)-n_2(t)d_{12}(t)]^2 - \alpha(t)e^{-\beta(t)}d_{11}(t)$$

Let us substitute in (4) the above expressions for the density derivatives $p(x,t)$ and coefficients $a_i(x,t), i=\overline{1,4}$. Comparing the terms of the same powers $(x_i(t) - n_i(t))$ and $(x_i(t)-n_i(t))(x_j(t)-n_j(t))$, $i, j = 1, 2$ in this equation, we can get the differential equation for $n_i(t)$, $d_{ij}(t)$, $i, j = 1, 2$ determination. In particular, if we compare the terms of $(x_1(t)-n_1(t))$, $(x_2(t)-n_2(t))$, then after the according transformations we get the ordinary differential equations set for the components of the vector $n(t)$

$$\begin{cases} n_1'(t) = (-\mu_1 - \mu_{01}(t))n_1(t) - \mu_{01}(t)n_2(t) + (1+\varepsilon(t))\mu_{01}(t) \\ n_2'(t) = \mu_1 n_1(t) - \mu_2 n_2(t) \end{cases} \tag{5}$$

In practice the function $K(t)$ is usually periodic and may possess the bounded values (in most cases sufficiently great). Therefore it is convenient to assign it in the following way

$$K(t) = \frac{q}{e\sin(bt)+d},\ q,e,d = const,\ \ q,e,d > 0,\ \ d > e$$

then $\varepsilon(t) = \frac{e\sin(bt)+d}{q} = \frac{e}{q}\sin(bt) + \frac{d}{q} = a\sin(bt) + c,\ a = \frac{e}{q},\ c = \frac{d}{q}$. Taking into consideration the type of the function $\mu_{01}(t)$, it is possible to find a general solution of the equations set (5) in the region A on the interval of time $[0, T/2]$, using the fundamental matrixes method [4]:

$$\begin{aligned} n_1^{1A}(t) = & \left[-\frac{(\lambda_1+\mu_2)(\mu_{01}+\mu_{01}c)}{(\lambda_2-\lambda_1)\lambda_1} - \frac{(\lambda_1+\mu_2)\mu_{01}ab}{(\lambda_1^2+b^2)(\lambda_2-\lambda_1)}\right]e^{\lambda_1 t} + \\ & + \left[\frac{(\lambda_2+\mu_2)(\mu_{01}+\mu_{01}c)}{(\lambda_2-\lambda_1)\lambda_2} + \frac{(\lambda_2+\mu_2)\mu_{01}ab}{(\lambda_2^2+b^2)(\lambda_2-\lambda_1)}\right]e^{\lambda_2 t} - \\ & - \frac{(\lambda_2+\mu_2)\mu_{01}a}{(\lambda_2^2+b^2)(\lambda_2-\lambda_1)}(\lambda_2\sin(bt)+b\cos(bt)) + \\ & + \frac{(\lambda_1+\mu_2)\mu_{01}a}{(\lambda_1^2+b^2)(\lambda_2-\lambda_1)}(\lambda_1\sin(bt)+b\cos(bt)) + \frac{(\mu_{01}+\mu_{01}c)\mu_2}{\lambda_1\lambda_2} \end{aligned} \tag{6}$$

$$\begin{aligned} n_2^{1A}(t) = & \left[-\frac{\mu_1(\mu_{01}+\mu_{01}c)}{(\lambda_2-\lambda_1)\lambda_1} - \frac{\mu_1\mu_{01}ab}{(\lambda_1^2+b^2)(\lambda_2-\lambda_1)}\right]e^{\lambda_1 t} + \\ & + \left[\frac{\mu_1(\mu_{01}+\mu_{01}c)}{(\lambda_2-\lambda_1)\lambda_2} + \frac{\mu_1\mu_{01}ab}{(\lambda_2^2+b^2)(\lambda_2-\lambda_1)}\right]e^{\lambda_2 t} - \\ & - \frac{\mu_1\mu_{01}a}{(\lambda_2^2+b^2)(\lambda_2-\lambda_1)}(\lambda_2\sin(bt)+b\cos(bt)) + \\ & + \frac{\mu_1\mu_{01}a}{(\lambda_1^2+b^2)(\lambda_2-\lambda_1)}(\lambda_1\sin(bt)+b\cos(bt)) + \frac{(\mu_{01}+\mu_{01}c)\mu_1}{\lambda_1\lambda_2} \end{aligned} \tag{7}$$

where:

$$\begin{aligned}&\lambda_1=-\frac{1}{2}\left(\mu_1+\mu_2+\mu_{01}+\sqrt{(\mu_1-\mu_2+\mu_{01})^2-4p_{12}\mu_1\mu_{01}}\right)\\&\lambda_2=-\lambda_1-\mu_1-\mu_2-\mu_{01}\end{aligned} \tag{8}$$

and also on the interval of time $(T/2,T]$:

$$\begin{aligned}n_1^{2A}(t)=&\left[(\lambda_2^1+\mu_2)e^{\lambda_2^1(t-T/2)}-(\lambda_1^1+\mu_2)e^{\lambda_1^1(t-T/2)}\right]\frac{n_1^{1A}(T/2)}{\lambda_2^1-\lambda_1^1}+\\&+\left[e^{\lambda_1^1(t-T/2)}-e^{\lambda_2^1(t-T/2)}\right]\frac{\mu_{01}^*n_2^{1A}(T/2)}{\lambda_2^1-\lambda_1^1}-\\&-\left[\frac{(\lambda_1^1+\mu_2)(\mu_{01}^*+\mu_{01}^*c)}{(\lambda_2^1-\lambda_1^1)\lambda_1^1}+\frac{(\lambda_1^1+\mu_2)\mu_{01}^*ab}{((\lambda_1^1)^2+b^2)(\lambda_2^1-\lambda_1^1)}\right]e^{\lambda_1^1(t-T/2)}+\\&+\left[\frac{(\lambda_2^1+\mu_2)(\mu_{01}^*+\mu_{01}^*c)}{(\lambda_2^1-\lambda_1^1)\lambda_2^1}+\frac{(\lambda_2^1+\mu_2)\mu_{01}^*ab}{((\lambda_2^1)^2+b^2)(\lambda_2^1-\lambda_1^1)}\right]e^{\lambda_2^1(t-T/2)}-\\&-\frac{(\lambda_2+\mu_2)\mu_{01}^*a}{((\lambda_2^1)^2+b^2)(\lambda_2^1-\lambda_1^1)}(\lambda_2^1\sin(b(t-T/2))+b\cos(b(t-T/2)))+\\&+\frac{(\lambda_1+\mu_2)\mu_{01}^*a}{((\lambda_1^1)^2+b^2)(\lambda_2^1-\lambda_1^1)}(\lambda_1^1\sin(b(t-T/2))+b\cos(b(t-T/2)))+\frac{(\mu_{01}^*+\mu_{01}^*c)\mu_2}{\lambda_1^1\lambda_2^1}\end{aligned} \tag{9}$$

$$\begin{aligned}n_2^{2A}(t)=&\left[e^{\lambda_2^1(t-T/2)}-e^{\lambda_1^1(t-T/2)}\right]\frac{p_{12}\mu_1n_1^{1A}(T/2)}{\lambda_2^1-\lambda_1^1}+\\&+\left[(\lambda_2^1+\mu_2)e^{\lambda_1^1(t-T/2)}-(\lambda_1^1+\mu_2)e^{\lambda_2^1(t-T/2)}\right]\frac{n_2^1(T/2)}{\lambda_2^1-\lambda_1^1}-\\&-\left[\frac{\mu_1(\mu_{01}^*+\mu_{01}^*c)}{(\lambda_2^1-\lambda_1^1)\lambda_1^1}+\frac{\mu_1\mu_{01}^*ab}{((\lambda_1^1)^2+b^2)(\lambda_2^1-\lambda_1^1)}\right]e^{\lambda_1^1(t-T/2)}+\\&+\left[\frac{\mu_1(\mu_{01}^*+\mu_{01}^*c)}{(\lambda_2^1-\lambda_1^1)\lambda_2^1}+\frac{\mu_1\mu_{01}^*ab}{((\lambda_2^1)^2+b^2)(\lambda_2^1-\lambda_1^1)}\right]e^{\lambda_2^1(t-T/2)}-\\&-\frac{\mu_1\mu_{01}^*a}{((\lambda_2^1)^2+b^2)(\lambda_2^1-\lambda_1^1)}(\lambda_2^1\sin(b(t-T/2))+b\cos(b(t-T/2)))+\\&+\frac{\mu_1\mu_{01}^*a}{((\lambda_1^1)^2+b^2)(\lambda_2^1-\lambda_1^1)}(\lambda_1^1\sin(b(t-T/2))+b\cos(b(t-T/2)))+\frac{(\mu_{01}^*+\mu_{01}^*c)\mu_1}{\lambda_1^1\lambda_2^1}\end{aligned} \tag{10}$$

where λ_1^1,λ_2^1 are computed in the same way as λ_1,λ_2 using formula (8) changing μ_{01} on μ_{01}^*.

3. On the solution of the optimization problem in the case of equitype claims

Note, that all found functions $n_i^{jA}(t), i, j = 1,2$ by formulae (6)-(10) have the following type

$$n(t) = me^{-\alpha t} + qe^{-\beta t} + h\sin(bt) + g\cos(bt) + s,\ \ \alpha > \beta > 0$$

with determined coefficients $m, q, h, g, s = const$, and do not depend on the number of company employees *n*. Obviously, in this case the functional $W(T, n)$ is linear increasing function of *n*, $W(T, n) = Dn + C$. Hence the solution of the problem (2) is a minimal *n* when on average there won't be queues on the stages of estimation and payment. I.e., the solution of the problem (2) reduces to determination of the minimal number *n*, which satisfies the conditions of this problem.

Since on the intervals of time $[0, T/2], (T/2, T]$ the function $\mu_{01}(t)$ possesses two different values, then optimal number *n* is different for these intervals. Let us denote these numbers n_1^*, n_2^*. First let us consider the work of an insurance company on the interval of time $[0, T/2]$. According to obtained results (6), (7) the average number of claims $N_i(t) = K(t)n_i^{1A}(t),\ i = 1,2$ at the moment of time *t* in the stages of estimation and payment does not depend on *n*, therefore, as it follows from constrains (2) and type of the functional $W(T, n)$

$$n_1^* = \left[1 + N_1^{1A}\right] + 1$$

where $[\ldots]$ is an integer part of the number in brackets

$$N_1^{1A} = \max_{t \in [0, T/2]} (K(t)n_1^{1A}(t))$$

Obviously, the average number of claims at any moment of time from the interval $[0, T/2]$ does not exceed $n^* - 1$ - the number of estimators working in an insurance company on this time interval. At the same time we should remember that the inequality $n_2^{1A}(t) \le l_2(t), t \in [0, T/2]$ has to be true. So it is necessary that

$$\max_{t \in [0, T/2]} (K(t)n_2^{1A}(t)) \le 1$$

At the moment of time $t = T/2$ the function $\mu_{01}(t)$ changes its value and becomes equal to μ_{01}^*. The problem (2) corresponds the case when the system is in the region *A* on the intervals of time $[0, T/2]$, $(T/2, T]$. At the same time as it follows from (9), (10) the values $N_i(t) = K(t)n_i^{2A}(t),\ i = 1,2$ do not depend on n on the interval of time $(T/2, T]$ as before and therefore

$$n_2^* = \left[1 + N_1^{2A}\right] + 1$$

where

$$N_1^{2A} = \max_{t \in (T/2, T]} (K(t) n_1^{2A}(t))$$

Also the following inequality should hold

$$\max_{t \in (T/2, T]} (K(t) n_2^{2A}(t)) \le 1$$

So the optimal number of estimators on the intervals $[0, T/2]$, $(T/2, T]$ equals to $n_1^* - 1 = \left[1 + N_1^{1A}\right]$, $n_2^* - 1 = \left[1 + N_1^{2A}\right]$ accordingly. Note, that in brackets of this ratios the ones are added because at computation N_1^{1A}, N_1^{2A} do not exceed one, since $n_1^{1A}(t)$, $n_1^{2A}(t)$ are computed accurate within terms of infinitesimal order $\varepsilon(t) = 1/K(t)$.

4. Model analysis for multi-type claims

Now we assume that an insurance company concludes with insurers contracts of $n-1$ types, i.e. at the some moment of time t from the considered interval of time $[0, T]$ the company concluded $K_i(t)$ contracts of the type i, $i = \overline{1, n-1}$, $\sum_{i=1}^{n-1} K_i(t) = K(t)$. For example it may be life, property, trucking industry, profession insurance and so on. The claims may be in the states C_i, $i = \overline{0,2}$, described above. Let m_i estimators be engaged in estimation of the type i claims, one cashier is engaged in payment. Type i claim transition probability from the state C_0 to the state C_1 on the interval of time $[t,\ t + \Delta t]$ equals to $\mu_{0i}(t)\Delta t + o(\Delta t)$, where $\mu_{0i}(t)$ is an intensity of such transition, the times of type i claim processing are distributed according to exponential rule with intensity μ_i, $r = \overline{1, n-1}$. Assume, that at some moment of time our system is in the state $k(t) = \left(k_0(t), k_1(t), ..., k_{n-1}(t), k_n(t)\right)$, if at this moment $k_i(t), i = \overline{1, n-1}$ claims of the type i are in the state C_1, and $k_n(t)$ claims are in the state C_2. Let D_i be a company's loss in a unit of time from one claim of the type i, when it is in the state C_1, $i = \overline{1, n-1}$, D_n be a company's loss from the claim in the state C_2, E_i be a salary of one estimator of the type i clams in a unit of time $i=\overline{1,n-1}$, E_n be a salary of a cashier, then as before the average company's loss on the interval of time $[0,T]$ may be described as before with a functional

$$W(T)=W(T,m_1,...,m_{n-1})=\frac{1}{T}\int_0^T\left[K(t)\sum_{i=1}^{n}(d_in_i(t)+E_il_i(t))\right]dt$$

where $n_i(t)=M\left\{\frac{k_i(t)}{K(t)}\right\}$, $l_i(t)=\frac{m_i}{K(t)}$, $i=\overline{1,n-1}$, $m_n=1$. Now we are interested in the following problem: to find the number of estimators m_i, $i=\overline{1,n-1}$, who have to work on different intervals of time so that the average number of type i claims in the estimation and payment stages at the moment of time t, i.e. $K(t)n_i(t)$ and $K(t)n_n(t)$, does not exceed m_i and 1 accordingly, $r=\overline{1,n-1}$, and the loss $W(T)$ is minimal:

$$\begin{cases} W(T)\to \min\limits_{m_i,\, i=\overline{1,n-1}} \\ K(t)n_i(t)\le m_i,\, i=\overline{1,n},\, t\in[0,T] \end{cases} \tag{11}$$

In this case, the probabilistic model of claims processing may be the queueing network, consisting of $n+1$ systems $S_0,S_1,S_2,...,S_n$, with service lines number $K,m_1,m_2,...,m_{n-1}\ge 1$, $m_n=1$ accordingly, and $1<<K(t)\le K$. Claims transition probabilities between the network's systems are $p_{0i}\ne 0$, $p_{in}=p_{n0}=1$, $i=\overline{1,n-1}$, $p_{ij}=0$ in other cases; at the moment of time t $K(t)$ claims are processed in the system; disciplines of claims processing in the network systems are FIFO. Note that as before the network is closed by the structure, but the number of claims processed in it depends on time.

It is determined, that the density of probabilities distribution $p(x,t)=$ $=p(x_1(t),x_2(t),...,x_n(t))$ of the vector $\frac{k(t)}{K(t)}=\left(\frac{k_1(t)}{K(t)},\frac{k_2(t)}{K(t)},...,\frac{k_n(t)}{K(t)}\right)$ accurate within $O(\varepsilon^2(t))$, where $\varepsilon(t)=l_n(t)$ satisfies the partial derivatives equation

$$\frac{\partial p(x,t)}{\partial t}=-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\big(A_i(x,t)p(x,t)\big)+\frac{\varepsilon(t)}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}\big(B_{ij}(x,t)p(x,t)\big)+ \\ +n\varepsilon(t)K'(t)p(x,t) \tag{12}$$

where $A_i(x,t)=\sum_{j=1}^{n}\mu_jq_{ji}\min(l_j(t),x_j(t))+\mu_{0i}(t)(1-\sum_{j=1}^{n}x_j(t));\ \mu_{0n}(t)=0$

$$B_{ij}(x,t)=\begin{cases}\sum_{j=1}^{n}\mu_j\min(l_j(t),x_j(t))+\mu_{0i}(t)(1-\sum_{j=1}^{n}x_j(t)),i=j\\ -2\mu_i\min(l_i(t),x_i(t)),i\neq j,j=n\\ 0,i\neq j,j=\overline{1,n-1}\end{cases}\qquad q_{ji}=\begin{cases}-1,j=i,j,i=\overline{1,n}\\ 1,j\neq i,i=n\\ 0,j\neq i,i=\overline{1,n-1}\end{cases}$$

When $K(t) = K$ this equation turns into Fokker-Planc-Kolmogorov equation for the vector x, hence we can use the Gaussian approximation for its solution.

5. The case of two types insurance contracts, $n=3$

Let's get seek the solution of the equation (12) as the probabilities distribution density of three-dimensional random quantity

$$p(x,t)=\frac{1}{\sqrt{(2\pi)^3}}\sqrt{|D(t)|}\exp\left[-\frac{1}{2}\sum_{i,j=1}^{3}(x_i(t)-n_i(t))d_{ij}(t)(x_j(t)-n_j(t))\right]$$

Further we should seek the solution of the optimization problem under the constraints $A_1=\{0\le n_i(t)\le l_i(t),i=\overline{1,3}\}$. It follows from the type of coefficients $B_{ij}(x,t)$ and definitions $l_i(t)$, $x_i(t)$, $i=1,2,3$, $\mu_{03}(t)$, and bounded density $p(x,t)$ that the part of the expression $\frac{\varepsilon(t)}{2}\sum_{i,j=1}^{3}\frac{\partial^2}{\partial x_i\partial x_j}\left(B_{ij}(x,t)p(x,t)\right)$, concerning the items of the type $\frac{\varepsilon(t)}{2}\frac{\partial^2}{\partial x_i\partial x_j}\left(\mu_i\min(l_i(t),x_i(t))\right)$ in the equation (12) vanishes, so in the region A_1 it may be written in the form:

$$\frac{\partial p(x,t)}{\partial t}=-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(A_i(x,t)p(x,t)\right)+\frac{\varepsilon(t)}{2}\mu_{01}(t)\frac{\partial^2}{\partial x_1^2}[(1-x_1(t)-x_2(t)-x_3(t))p(x,t)]+$$
$$+\frac{\varepsilon(t)}{2}\mu_{02}(t)\frac{\partial^2}{\partial x_2^2}[(1-x_1(t)-x_2(t)-x_3(t))p(x,t)]+3\varepsilon(t)K'(t)p(x,t)$$

Making use of several obvious transformations for the partial derivatives on the right side we get

$$\frac{\partial p(x,t)}{\partial t} = [\mu_1 \frac{\partial \min(l_1(t), x_1(t))}{\partial x_1} + \mu_2 \frac{\partial \min(l_2(t), x_2(t))}{\partial x_2} +$$

$$+ \mu_3 \frac{\partial \min(l_3(t), x_3(t))}{\partial x_3} + \mu_{01}(t) + \mu_{02}(t) + 3\varepsilon(t)K'(t)]p(x,t) +$$

$$+[\mu_1 \min(l_1(t), x_1(t)) - \mu_{01}(t)(1 - x_1(t) - x_2(t) - x_3(t)) - \varepsilon(t)\mu_{01}(t)] \frac{\partial p(x,t)}{\partial x_1} +$$

$$+[\mu_2 \min(l_2(t), x_2(t)) - \mu_{02}(t)(1 - x_1(t) - x_2(t) - x_3(t)) - \varepsilon(t)\mu_{02}(t)] \frac{\partial p(x,t)}{\partial x_2} + \quad (13)$$

$$+[\mu_3 \min(l_3(t), x_3(t)) - \mu_2 \min(l_2(t), x_2(t)) - \mu_1 \min(l_1(t), x_1(t))] \frac{\partial p(x,t)}{\partial x_3} +$$

$$+ \frac{\varepsilon(t)}{2} \mu_{01}(t)(1 - x_1(t) - x_2(t) - x_3(t)) \frac{\partial^2 p(x,t)}{\partial x_1^2} +$$

$$+ \frac{\varepsilon(t)}{2} \mu_{02}(t)(1 - x_1(t) - x_2(t) - x_3(t)) \frac{\partial^2 p(x,t)}{\partial x_2^2}$$

Using an expansion of the coefficients of the equation (13) in Taylor series in the neighborhood of the point $(n_1(t), n_2(t), n_3(t))$ and not going beyond the first expansion terms we get

$$a_1(n,t) = \mu_1 + \mu_2 + \mu_3 + \mu_{01}(t) + \mu_{02}(t) + 3\varepsilon(t)K'(t)$$

$$a_2(n,t) = (\mu_1 + \mu_{01}(t))n_1(t) + \mu_{01}(t)n_2(t) - \mu_{01}(t)(\varepsilon(t) + 1)$$

$$a_3(n,t) = \mu_{01}(t)n_1(t) + (\mu_2 + \mu_{02}(t))n_2(t) - \mu_{02}(t)(\varepsilon(t) + 1)$$

$$a_4(n,t) = \mu_3 n_3(t) - \mu_2 n_2(t) - \mu_1 n_1(t)$$

$$a_4(n,t) = \frac{\varepsilon(t)}{2} \mu_{01}(t)(1 - n_1(t) - n_2(t) - n_3(t))$$

$$a_5(n,t) = \frac{\varepsilon(t)}{2} \mu_{02}(t)(1 - n_1(t) - n_2(t) - n_3(t))$$

Further we can find the derivatives of the function

$$p(x,t) = p(x_1(t), x_2(t), x_3(t)) =$$

$$= \frac{1}{\sqrt{(2\pi)^3}} \sqrt{\det D(t)} \exp[-\frac{1}{2}[(x_1(t) - n_1(t))^2 d_{11}(t) + (x_2(t) - n_2(t))^2 d_{22}(t) +$$

$$+ (x_3(t) - n_3(t))^2 d_{33}(t) + 2(x_1(t) - n_1(t))(x_2(t) - n_2(t))d_{12}(t) + 2(x_1(t) - n_1(t)) \times$$

$$\times (x_3(t) - n_3(t))d_{13}(t) + 2(x_3(t) - n_3(t))(x_2(t) - n_2(t))d_{23}(t)] = \alpha(t)e^{-\beta(t)}$$

where

$$\det D(t) = d_{11}(t)d_{22}(t)d_{33}(t) + d_{12}(t)d_{13}(t)d_{23}(t) + d_{12}(t)d_{23}(t)d_{13}(t) - \\ - d_{11}(t)d_{23}^2(t) - d_{33}(t)d_{12}^2(t) - d_{11}(t)d_{13}^2(t)$$

Substituting the obtained expressions for the derivatives of the density $p(x,t)$ and coefficients $a_i(x,t)$, $i=\overline{1,5}$, in (13) and comparing the terms of the same powers $(x_i(t)-n_i(t))$ and $(x_i(t)-n_i(t))(x_j(t)-n_j(t))$, i,j = 1,2,3, we can get the differential equations for $n_i(t)$, $d_{ij}(t)$, i, j = 1,2 determination. For example, comparing the terms of $(x_1(t)-n_1(t))$, $(x_2(t)-n_2(t))$, $(x_3(t)-n_3(t))$, after the according transformations we get the ordinary differential equations set for the components of the vector $n(t)$:

$$\begin{cases} \dfrac{dn_1(t)}{dt} = -\mu_1 n_1(t) + \mu_{01}(t)(1+\varepsilon(t) - \sum_{i=1}^{3} n_i(t)) \\ \dfrac{dn_2(t)}{dt} = -\mu_2 n_2(t) + \mu_{02}(t)(1+\varepsilon(t) - \sum_{i=1}^{3} n_i(t)) \\ \dfrac{dn_3(t)}{dt} = \mu_1 n_1(t) + \mu_2 n_2(t) - \mu_3 n_3(t) \end{cases} \tag{14}$$

Assume, that each of two considered types claims entry intensity $\mu_{0i}(t)$, i = 1,2 are piecewise constant functions

$$\mu_{01}(t) = \begin{cases} \mu_{01}^*, t\in[0,T/2] \\ \mu_{01}^{**}, t\in(T/2,T] \end{cases} \quad \mu_{02}(t) = \begin{cases} \mu_{02}^*, t\in[0,T/2] \\ \mu_{02}^{**}, t\in(T/2,T] \end{cases}$$

Then using the fundamental matrixes method we can find the general solution of the equation set (14) $(n_1^{1A}(t), n_2^{1A}(t), n_3^{1A}(t))$ in the region A_1 on the interval of time $[0,T/2]$. For example, $n_1^{1A}(t)$ has the type:

$$n_1^{1A}(t) = \frac{e^{\lambda_1 t}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\left[\left(\lambda_1^2\mu_{01}^* + \lambda_1(\mu_2\mu_{01}^* + \mu_3\mu_{01}^*) + \mu_2\mu_3\mu_{01}^*\right)\left(\frac{1+c}{\lambda_1} + \frac{ab}{\lambda_1^2+b^2}\right)\right] + \\ + \frac{e^{\lambda_2 t}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)}\left[\left(-\lambda_2^2\mu_{01}^* - \lambda_2(\mu_2\mu_{01}^* + \mu_3\mu_{01}^*) - \mu_2\mu_3\mu_{01}^*\right)\left(\frac{1+c}{\lambda_2} + \frac{ab}{\lambda_2^2+b^2}\right)\right] + \\ + \frac{e^{\lambda_3 t}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\left[\left(\lambda_3^2\mu_{01}^* + \lambda_3(\mu_2\mu_{01}^* + \mu_3\mu_{01}^*) + \mu_2\mu_3\mu_{01}^*\right)\left(\frac{1+c}{\lambda_3} + \frac{ab}{\lambda_3^2+b^2}\right)\right] -$$

$$-a\left(\lambda_1^2\mu_{01}^* + \lambda_1(\mu_2\mu_{01}^* + \mu_3\mu_{01}^*) + \mu_2\mu_3\mu_{01}^*\right)\frac{\lambda_1\sin(bt) + b\cos(bt)}{(\lambda_1^2 + b^2)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \tag{15}$$

$$+ a\left(-\lambda_2^2\mu_{01}^* - \lambda_2(\mu_2\mu_{01}^* + \mu_3\mu_{01}^*) - \mu_2\mu_3\mu_{01}^*\right)\frac{\lambda_2\sin(bt) + b\cos(bt)}{(\lambda_2^2 + b^2)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} -$$

$$- a\left(\lambda_3^2\mu_{01}^* + \lambda_3(\mu_2\mu_{01}^* + \mu_3\mu_{01}^*) + \mu_2\mu_3\mu_{01}^*\right)\frac{\lambda_3\sin(bt) + b\cos(bt)}{(\lambda_3^2 + b^2)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} - \frac{\mu_2\mu_3\mu_{01}^*(1 + c)}{\lambda_1\lambda_2\lambda_3}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the characteristic equation of the equation set (14). $n_2^{1A}(t), n_3^{1A}(t)$ have the same type and corresponding expressions on the interval of time $(T/2, T]$ – $n_i^{2A}(t)$, $i = \overline{1,3}$.

6. On a solution of the optimization problem for the claims of two types

Let's go back to the original problem (11) when $n = 3$. Since all the functions $n_i^{jA}(t)$, $i = 1,2,3, j = 1,2$, have the type

$$n(t) = me^{-\alpha t} + fe^{-\beta t} + ke^{-\gamma t} + h\sin(bt) + g\cos(bt) + s,\ \alpha > \beta > \gamma > 0$$

with determined coefficients $m, n, k, f, g, s = const$, and do not depend on the number of company's employees m_i, i = 1,2, then in this case the functional $W(T)$ is linear increasing function of m_i, i = 1,2, $W(T, m_1, m_2) = Dm_1 + Bm_2 + C,\ D, B > 0$. Hence, the solution of the problem (11) is a minimal m_i, i = 1,2 when on average there won't be queues on the stages of estimation and payment.

Let m_1^*, m_2^* be optimal numbers of the claims estimators on the interval of time $[0, T]$. According to (15) the average number of claims at the moment of time t in the estimation and payment stages $N_i^{1A} = K(t)n_i^{1A}(t)$, $i = \overline{1,3}$, does not depend on m_1, m_2, therefore, as it follows from the constrains of the problem (11) and the type of the functional $W(T, m_1, m_2)$

$$m_1^* = [1 + N_1^{1A}],\ m_2^* = [1 + N_2^{1A}]$$

where […] is an integer part of the number in brackets, $N_i^{1A} = \max\limits_{t\in[0,T/2]}(K(t)n_i^{1A}(t))$, $i = 1,2$. At the same time we should remember that the inequality $n_3^{1A}(t) \le l_3(t), t \in [0, T/2]$ has to be true. For the correctness of this inequality we need that

$$\max_{t\in[0,T/2]}(K(t)n_3^{1A}(t))\le 1$$

So the optimal number of the estimators on the interval $[0,T/2]$ equals to $m_1^*-1=[1+N_1^{1A}]$, $m_2^*-1=[1+N_2^{1A}]$ correspondingly. In the same way it is possible to find the optimal numbers of claims estimators on the interval $(T/2,T]$.

7. Problems

The considering research on the claims processing models in insurance companies enable us to raise some new problems which are important from the theoretical and practical points of view.

1. In the general case for an arbitrary n for the solution of the equation (12) it is also possible to use the approximation method called Gaussian approximation. Then we get the following differential equations set:

$$\begin{cases} \dfrac{dn_j(t)}{dt} = -\mu_1 n_j(t) + \mu_{0j}(t)(1+\varepsilon(t)-\sum\limits_{i=1}^{n} n_i(t)), \quad j=\overline{1,n-1} \\ \dfrac{dn_n(t)}{dt} = \mu_1 n_1(t) + \mu_2 n_2(t) + \ldots + \mu_{n-1} n_{n-1}(t) - \mu_n n_n(t) \end{cases}$$

   It is necessary to work out the analytical methods for the solutions of such equations sets or at least to find out for what type of functions $\mu_{0j}(t)$, $j=\overline{1,n-1}$, it is possible to find analytical solutions.

2. It is necessary to work out the methods for characteristic models when the times of claims processing on different stages are distributed according to the rules different from exponential one. Such models describe a real situation more adequately.
3. The research on the models when the general number o claims $K(t)$ is constant or is a function of time. From the practical point of view it is interesting to investigate a case when $K(t)$ is a random process, so it is also a topical problem.

References

[1] Matalytski M., Astahov A., Investigation of one probabilistic model in insurance, Queues: flows, systems, networks, Proceeding of BWWQT - 99, BSU, Minsk 1999, 98-103.

[2] Matalytski M., Astahov A., Analysis of network model in one insurance task, Mathematical methods in finance and economics, BSU, Minsk 2000, 42-47.

[3] Romaniuk T., Matalytski M., Analysis of probabilistic model of heterogeneous claims processing in insurance company, Novyje matematicheskije metody i komp'juternyje tehnologii v proektirovanii, proizvodstve i nauchnyh issledovanijah, Gomel, GGU, 2001, 52-54 (in Russian).

[4] Parajev Y., Introduction statistical dynamics of control processes, Sov. Radio, Moscow 1976.