SHAPE SENSITIVITY ANALYSIS OF TEMPERATURE DISTRIBUTION IN HEATING TISSUE

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Abstract. In the paper the numerical analysis of heat transfer process proceeding in domain of biological tissue is presented (2D problem). The temperature distribution results from the action of external heat source emanating the boundary heat flux. The sensitivity analysis of this process with respect to the geometrical parameters of biological tissue is considered. The explicit differentiation method basing on the material derivative approach is used. On the stage of numerical computations the boundary element method is applied. In the final part of the paper the results obtained are shown.

1. Formulation of the problem

The temperature field in domain of biological tissue (Fig. 1) subjected to the action of boundary heat flux q_0 is described by the following equation [1] and conditions:

$$\begin{aligned}
&(x, y) \in \Omega : \quad \lambda \frac{\partial^2 T(x, y)}{\partial x^2} + \lambda \frac{\partial^2 T(x, y)}{\partial y^2} + k\left[T_B - T(x, y)\right] + Q_m = 0 \\
&(x, y) \in \Gamma_1 : \quad T(x, y) = T_b \\
&(x, y) \in \Gamma_2 : \quad q(x, y) = -\lambda\, \mathbf{n} \cdot \nabla T(x, y) = 0 \\
&(x, y) \in \Gamma_0 : \quad q(x, y) = -\lambda\, \mathbf{n} \cdot \nabla T(x, y) = q_0
\end{aligned} \tag{1}$$

where λ [W/(mK)] is the thermal conductivity, $k = c_B G_B$ (c_B [J/(m^{3}K)] is the volumetric specific heat of blood, G_B [m^3 blood/s/m^3 tissue] is the perfusion rate), T_B is the blood temperature, Q_m [W/m^3] is the metabolic heat source, T is the temperature, x, y are the geometrical co-ordinates. The domain Ω is the square of dimensions $L \times L$, while the boundary heat flux q_0 changes according to the formula

$$q_0(y) = q_m \frac{L^2 - y^2}{L^2} \tag{2}$$

where $q_0(0) = q_m$ is the maximum value of this heat flux.

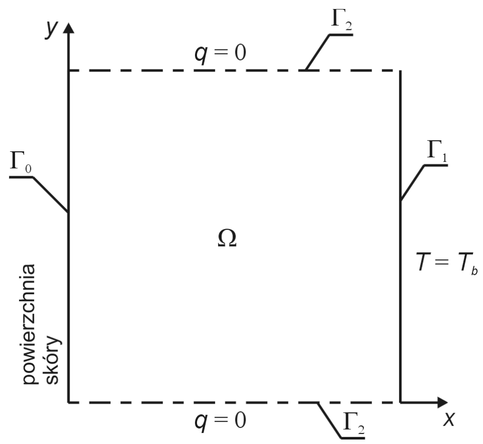

Fig. 1. Domain considered

We assume that the shape parameter b corresponds to the half of square diagonal, this means $b = L\sqrt{2}/2$. The aim of investigations is to estimate the changes of temperature due to the expansion (or contraction) of the domain considered.

2. Explicit differentiation method

Using the concept of material derivative we can write [2, 3]

$$\frac{\mathrm{D}T}{\mathrm{D}b} = \frac{\partial T}{\partial b} + \frac{\partial T}{\partial x} v_x + \frac{\partial T}{\partial y} v_y \tag{3}$$

where b is the shape parameter, $v_x = v_x(x, y, b)$ and $v_y = v_y(x, y, b)$ are the velocities associated with design parameter b.
If the direct approach of sensitivity method is applied [2, 3] then the equation (1) are differentiated with respect to shape parameter b, and then:

$$\begin{aligned}
&(x, y) \in \Omega: \quad \lambda \frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial^2 T}{\partial x^2}\right) + \lambda \frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial^2 T}{\partial y^2}\right) - k\frac{\mathrm{D}T}{\mathrm{D}b} = 0\\
&(x, y) \in \Gamma_1: \quad \frac{\mathrm{D}T}{\mathrm{D}b} = \frac{\mathrm{D}T_b}{\mathrm{D}b} = 0\\
&(x, y) \in \Gamma_2: \quad \frac{\mathrm{D}q}{\mathrm{D}b} = -\lambda \frac{\mathrm{D}(\mathbf{n} \cdot \nabla T)}{\mathrm{D}b} = 0\\
&(x, y) \in \Gamma_0: \quad \frac{\mathrm{D}q}{\mathrm{D}b} = -\lambda \frac{\mathrm{D}(\mathbf{n} \cdot \nabla T)}{\mathrm{D}b} = \frac{\mathrm{D}q_0}{\mathrm{D}b}
\end{aligned} \tag{4}$$

On the basis of definition (3) the following formulas can be derived [4]:

$$\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial^2 T}{\partial x^2}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right)-2\,\frac{\partial^2 T}{\partial x^2}\frac{\partial v_x}{\partial x}-\frac{\partial T}{\partial x}\frac{\partial^2 v_x}{\partial x^2}-2\,\frac{\partial^2 T}{\partial x\partial y}\frac{\partial v_y}{\partial x}-\frac{\partial T}{\partial y}\frac{\partial^2 v_y}{\partial x^2} \tag{5}$$

and:

$$\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial^2 T}{\partial y^2}\right)=\frac{\partial^2}{\partial y^2}\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right)-2\,\frac{\partial^2 T}{\partial y^2}\frac{\partial v_y}{\partial y}-\frac{\partial T}{\partial x}\frac{\partial^2 v_x}{\partial y^2}-2\,\frac{\partial^2 T}{\partial x\partial y}\frac{\partial v_x}{\partial y}-\frac{\partial T}{\partial y}\frac{\partial^2 v_y}{\partial y^2} \tag{6}$$

while

$$\frac{\mathrm{D}\left(\mathbf{n}\cdot\nabla T\right)}{\mathrm{D}b}=\mathbf{n}\cdot\nabla\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right)+\mathbf{n}\cdot\nabla T\cdot\mathbf{n}\cdot\nabla v\cdot\mathbf{n}^{\mathrm{T}}-\mathbf{n}\cdot\left[\nabla v+\left(\nabla v\right)^{\mathrm{T}}\right]\cdot\nabla T \tag{7}$$

where

$$\nabla v=\begin{bmatrix}\dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_y}{\partial x}\\ \dfrac{\partial v_x}{\partial y} & \dfrac{\partial v_y}{\partial y}\end{bmatrix} \tag{8}$$

In the case of expansion or contraction of the domain considered it is assumed that the velocities $v_x = x/b$, $v_y = y/b$. So

$$\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial^2 T}{\partial x^2}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right)-\frac{2}{b}\frac{\partial^2 T}{\partial x^2} \tag{9}$$

and

$$\frac{\mathrm{D}}{\mathrm{D}b}\left(\frac{\partial^2 T}{\partial y^2}\right)=\frac{\partial^2}{\partial y^2}\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right)-\frac{2}{b}\frac{\partial^2 T}{\partial y^2} \tag{10}$$

while

$$\frac{\mathrm{D}\left(\mathbf{n}\cdot\nabla T\right)}{\mathrm{D}b}=\mathbf{n}\cdot\nabla\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right)-\frac{1}{b}\,\mathbf{n}\cdot\nabla T \tag{11}$$

Introducing (9), (10), (11) into (4) and taking into account that from equation (1) results that

$$\lambda \frac{\partial^2 T}{\partial x^2} + \lambda \frac{\partial^2 T}{\partial y^2} = k\left(T - T_B\right) - Q_m \tag{12}$$

one obtains:

$$\begin{aligned}
&(x, y) \in \Omega: \quad \lambda \frac{\partial^2}{\partial x^2}\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right) + \lambda \frac{\partial^2}{\partial y^2}\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right) - k\frac{\mathrm{D}T}{\mathrm{D}b} - \frac{2}{b}T + \frac{2}{b}Q = 0 \\
&(x, y) \in \Gamma_1: \quad \frac{\mathrm{D}T}{\mathrm{D}b} = 0 \\
&(x, y) \in \Gamma_2: \quad -\lambda\, \mathbf{n} \cdot \nabla\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right) = 0 \\
&(x, y) \in \Gamma_0: \quad -\lambda\, \mathbf{n} \cdot \nabla\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right) = \frac{q_0}{b}
\end{aligned} \tag{13}$$

where $Q = kT_B + Q_m$.
Finally:

$$\begin{aligned}
&(x, y) \in \Omega: \quad \lambda \frac{\partial^2 U}{\partial x^2} + \lambda \frac{\partial^2 U}{\partial y^2} - kU - \frac{2}{b}kT + \frac{2}{b}Q = 0 \\
&(x, y) \in \Gamma_1: \quad U = 0 \\
&(x, y) \in \Gamma_2: \quad W = 0 \\
&(x, y) \in \Gamma_0: \quad W = \frac{q_0}{b}
\end{aligned} \tag{14}$$

where $U = \mathrm{D}T/\mathrm{D}b$ is the sensitivity function and

$$W = -\lambda\, \mathbf{n} \cdot \nabla\left(\frac{\mathrm{D}T}{\mathrm{D}b}\right) = -\lambda\, \mathbf{n} \cdot \nabla U \tag{15}$$

3. Boundary element method

In order to solve the basic problem (1) and additional problem (14) connected with the shape sensitivity analysis the boundary element method has been applied. So, we consider the following equation

$$\lambda \frac{\partial^2 F(x, y)}{\partial x^2} + \lambda \frac{\partial^2 F(x, y)}{\partial y^2} + S(x, y) = 0 \tag{16}$$

where for the problem (1): $F(x, y) = T(x, y)$ and $S(x, y) = -kT(x, y) + Q$, while for the problem (14): $F(x, y) = U(x, y)$ and $S(x, y) = -kU(x, y) - 2kT(x, y)/b + 2Q/b$. The boundary integral equation for equation (16) is following [4, 5]:

$$\begin{gathered}(\xi, \eta) \in \Gamma: \quad B(\xi, \eta) F(\xi, \eta) + \int_\Gamma J(x, y) F^*(\xi, \eta, x, y) \, \mathrm{d}\Gamma = \\ \int_\Gamma F(x, y) J^*(\xi, \eta, x, y) \, \mathrm{d}\Gamma + \iint_\Omega S(x, y) F^*(\xi, \eta, x, y) \, \mathrm{d}\Omega\end{gathered} \tag{17}$$

where $B(\xi, \eta) \in (0, 1)$ is the coefficient connected with the local shape of boundary, (ξ, η) is the observation point, $F^*(\xi, \eta, x, y)$ is the fundamental solution

$$F^*(\xi, \eta, x, y) = \frac{1}{2\pi\lambda} \ln \frac{1}{r} \tag{18}$$

where r is the distance between the points (ξ, η) and (x, y)

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2} \tag{19}$$

Function $J^*(\xi, \eta, x, y)$ is defined as follows

$$J^*(\xi, \eta, x, y) = -\lambda \, \mathbf{n} \cdot \nabla F^*(\xi, \eta, x, y) \tag{20}$$

and it can be calculated analytically

$$J^*(\xi, \eta, x, y) = \frac{d}{2\pi r^2} \tag{21}$$

where

$$d = (x-\xi)\cos\alpha + (y-\eta)\cos\beta \tag{22}$$

In equation (17): $J(x, y) = -\lambda \, \mathbf{n} \cdot \nabla F(x, y)$.
In order to solve the equation (17) the boundary is divided into N boundary elements and the interior is divided into L internal cells. For constant boundary elements we assume that

$$(x, y) \in \Gamma_j: \begin{cases} F(x, y) = F(x_j, y_j) = F_j \\ J(x, y) = J(x_j, y_j) = J_j \end{cases} \tag{23}$$

and for constant internal cells

$$(x, y) \in \Omega_l: \quad S(x, y) = S(x_l, y_l) = S_l \tag{24}$$

In this case the following approximation of equation (17) is obtained

$$\frac{1}{2}F_i + \sum_{j=1}^{N} J_j \int_{\Gamma_j} F^*\left(\xi_i, \eta_i, x, y\right) \mathrm{d}\Gamma_j = \sum_{j=1}^{N} F_j \int_{\Gamma_j} J^*\left(\xi_i, \eta_i, x, y\right) \mathrm{d}\Gamma_j + \sum_{l=1}^{L} S_l \iint_{\Omega_l} F^*\left(\xi_i, \eta_i, x, y\right) \mathrm{d}\Omega_l \tag{25}$$

where (ξ_i, η_i) denotes the boundary node and $i = 1, 2, \ldots, N$.
The system of equations (25) can be written in the form

$$\sum_{j=1}^{N} G_{ij} J_j = \sum_{j=1}^{N} H_{ij} F_j + \sum_{l=1}^{L} P_{il} S_l \tag{26}$$

where

$$G_{ij} = \int_{\Gamma_j} F^*\left(\xi_i, \eta_i, x, y\right) \mathrm{d}\Gamma_j \tag{27}$$

and

$$H_{ij} = \begin{cases} \int_{\Gamma_j} J^*\left(\xi_i, \eta_i, x, y\right) \mathrm{d}\Gamma_j, & i \neq j \\ -\dfrac{1}{2}, & i = j \end{cases} \tag{28}$$

while

$$P_{il} = \iint_{\Omega_l} F^*\left(\xi_i, \eta_i, x, y\right) \mathrm{d}\Omega_l \tag{29}$$

After solving the system of equations (26) the boundary values of functions F_j and J_j are known. The values F_i at the internal points $(\xi_i, \eta_i) \in \Omega$ are calculated using the formula

$$F_i = \sum_{j=1}^{N} H_{ij} F_j - \sum_{j=1}^{N} G_{ij} J_j + \sum_{l=1}^{L} P_{il} S_l \tag{30}$$

It should be pointed out that in the case when the source function S is a function of T or U the basic algorithm should be supplemented by adequate iterative procedure [4].

4. Example of computations

The square domain of dimensions 0.02×0.02 m has been considered (Fig. 1). The following data have been assumed: λ = 0.75 W/(mK), G_B = 0.0005 1/s, c_B = = 3.9942 · 10^6 J/(m^3 K) (k = 1998.1), T_B = 37°C, Q_m = 245 W/m^3. On the boundary Γ_1 the temperature T_b = 37°C has been accepted, on the surface Γ_0 of biological tissue the boundary condition (2) has been assumed, where q_m = –2000 W/m^2. The boundary has been divided into N = 60 constant boundary elements, the interior has been divided into L = 225 constant internal cells (squares).

In Figure 2 the temperature distribution in the tissue domain is shown. Figure 3 illustrates the distribution of sensitivity function U.

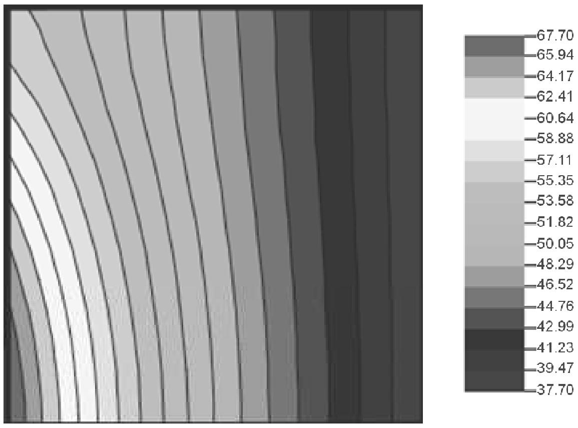

Fig. 2. Temperature distribution

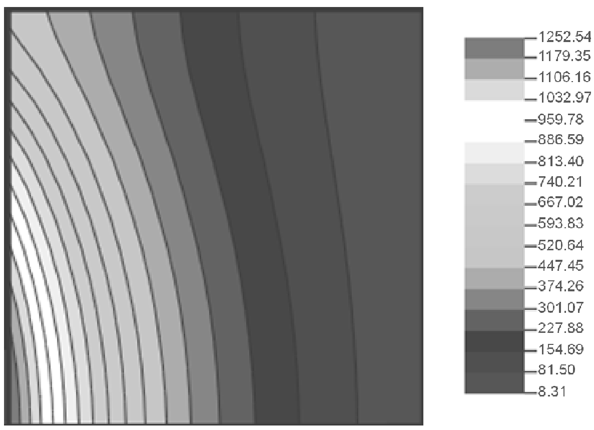

Fig. 3. Distribution of function U

Function T can be expanded into Taylor's series taking into account two components

$$T\left(x, y, b + \Delta b\right) = T\left(x, y, b\right) + U\left(x, y, b\right)\Delta b \tag{31}$$

In Figure 4 the temperature distribution obtained from equation (31) under the assumption that $\Delta b = 0.1b$ is shown.

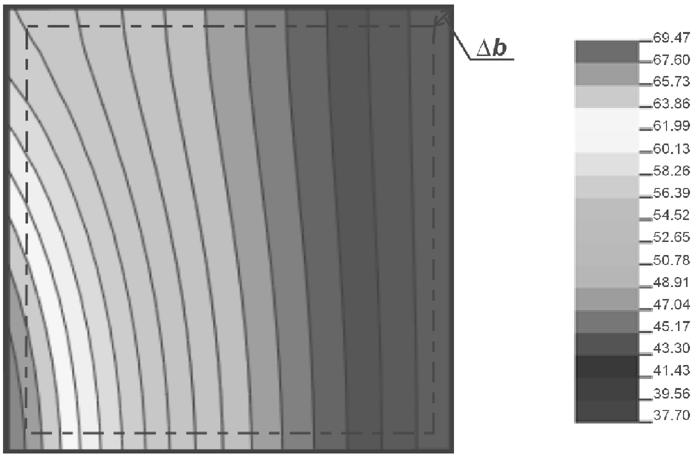

Fig. 4. Temperature distribution for $b + \Delta b$

Summing up, the shape sensitivity analysis allows, among others, to estimate the changes of temperature in the case when the geometry of the domain considered is changed.

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