TEMPERATURE DISTRIBUTION IN AN ANNULAR PLATE WITH A MOVING DISCRETE HEAT GENERATION SOURCE

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Abstract. The problem of heat conduction in an annular plate is considered. The plate is subjected to the activity of the point heat source which moves with constant angular velocity on the plate surface along a concentric circular trajectory. The solution of the problem is obtained by using the Green's function.

Introduction

Several authors have used Green's functions to solve the heat conduction problems [1-4]. In paper [1] the steady-state temperature distribution in circular plates and spheres with discrete internal heat generation sources has been determined. The temperature distribution was obtained from Green's function by using the method of images. A solution of the steady-state heat conduction problem in a solid cylinder for a variety of boundary conditions by using the Green's function method is presented in paper [2]. Influence functions appropriate for the boundary-element method are constructed with the Green's functions to describe the temperature field of a cylinder heated by a specified heat flux over a portion of one face. The use of modified Green's function in unsteady heat transfer is considered in paper [3].

In paper [5] authors study the thermoelastic problem of a thin circular plate. The plate is subjected to a partially distributed and axisymmetric heat supply on the curved surface. The authors develop the analysis for the temperature field by introducing the methods of the finite Fourier and the finite Hankel transform. The heat conduction problem in a circular thin plate subjected to the activity of a heat source is presented in paper [6]. The solution of the problem in analytical form is obtained by using the Green's function method.

In this paper, an analytical solution to the heat conduction problem in an annular plate is presented. The plate is heated by a heat source which moves on the plate surface along a concentric circular trajectory with constant angular velocity. The temperature distribution of the plate in an analytical form was obtained by using a time dependent Green's function.

1. Problem formulation

Consider temperature in the annular plate of thickness h, inner radius a and outer radius b (Fig. 1). This plate is heated by a heat source which moves on the plate surface along a concentric circular trajectory at radius r_0 with constant angular velocity ω.

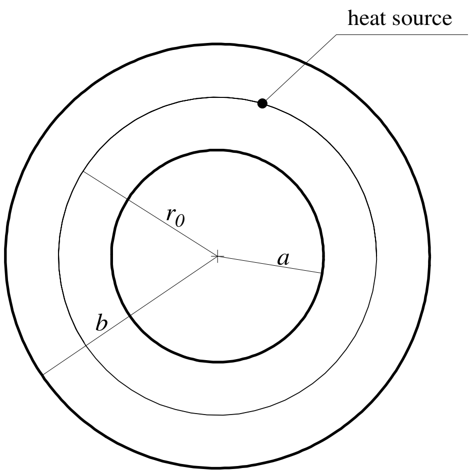

Fig. 1. A schematic of an annular plate with a heat source

The heat conduction equation in cylindrical coordinates has the form [4]:

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2}+\frac{1}{k}q(r,\phi,z,t)=\frac{1}{a}\frac{\partial T}{\partial t} \tag{1}$$

where: $T(r,\phi,z,t)$ - temperature of the plate at point (r,ϕ,z) at time t, k - thermal conductivity, a - thermal diffusivity and $q(r,\phi,z,t)$ represents an energy generation term.

The heat generation term is assumed in the form

$$q(r,\phi,z,t)=\theta\,\delta(r-r_0)\,\delta(\phi-\varphi(t))\,\delta(z-h) \tag{2}$$

where θ characterises the stream of the heat, $\delta(\ )$ is the Dirac delta function, $\varphi(t)$ is the function describing the movement of the heat source

$$\varphi(t)=\omega t \tag{3}$$

The temperature distribution in the annular plate is obtained as a solution of the equation (1) with the following initial and boundary conditions:

$$T(r,\phi,z,0)=0 \tag{4}$$

$$T\big|_{r=a} = T_1\,, \quad T\big|_{r=b} = T_2 \tag{5}$$

$$k\frac{\partial T}{\partial z}(r,\phi,h,t) = \alpha_0[T_0 - T(r,\phi,h,t)] \tag{6}$$

$$k\frac{\partial T}{\partial z}(r,\phi,0,t) = -\alpha_0[T_0 - T(r,\phi,0,t)] \tag{7}$$

where α_0 is the heat transfer coefficient, T_0 is the known temperature of the surrounding medium, T_1, T_2 are the known temperatures on boundaries of the plate.

2. Solution of the problem

The solution of the problem in an analytical form is obtained by using the prop erties of the Green's function (GF). The GF for the heat conduction problem describes the temperature distribution induced by the temporary, local energy impulse. The GF function is a solution to the differential equation [4]:

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} + \frac{1}{r^2}\frac{\partial^2 G}{\partial \phi^2} + \frac{\partial^2 G}{\partial z^2} - \frac{1}{a}\frac{\partial G}{\partial t} = \frac{\delta(r-\rho)\delta(\phi-\phi')\delta(z-\zeta)\delta(t-\tau)}{r} \tag{8}$$

Moreover, the Green's function satisfies the initial and homogeneous boundary conditions analogous to conditions (4)-(7).

The GF for the considered heat conduction problem may be written in the form of a series:

$$G(r,\phi,z,t) = \sum_{m=-\infty}^{\infty} g_m(r,z,t)\cos m(\phi-\phi') \tag{9}$$

Substituting the series (9) into equation (8) and using the expansion [7]

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\cos m(\phi-\phi') \tag{10}$$

the differential equation for the functions $g_m(r,z,t)$ is obtained

$$\frac{\partial^2 g_m}{\partial r^2} + \frac{1}{r}\frac{\partial g_m}{\partial r} + \frac{\partial^2 g_m}{\partial z^2} - \frac{m^2}{r^2}g_m - \frac{1}{a}\frac{\partial g_m}{\partial t} = \frac{\delta(r-\rho)\,\delta(z-\zeta)\delta(t-\tau)}{2\pi r} \tag{11}$$

Using (9) in boundary and initial conditions, we have

$$g_m(r,z,0) = 0\,, \quad g_m\big|_{r=a} = 0 \quad g_m\big|_{r=b} = 0 \tag{12}$$

$$\left.\left(\frac{\partial g_m}{\partial z}-\mu_0\, g_m\right)\right|_{z=0}=0\,,\quad \left.\left(\frac{\partial g_m}{\partial z}+\mu_0 g_m\right)\right|_{z=h}=0 \tag{13}$$

where $\mu_0=\dfrac{\alpha_0}{a}$.

The solution of the initial-boundary problem (11)-(13) can be presented in the form of a series

$$g_m(r,z,t)=\sum_{n=1}^{\infty}\Gamma_{mn}(r,t)\psi_n(z) \tag{14}$$

where $\psi_n(z)$ are eigenfunctions of the following boundary problem

$$\frac{\partial^2\psi_n}{\partial z^2}+\beta^2\psi_n(z)=0 \tag{15}$$

$$\left.\left(\frac{d\,\psi_n}{d\,z}-\mu_0\,\psi_n\right)\right|_{z=0}=0,\quad \left.\left(\frac{d\,\psi_n}{d\,z}+\mu_0\,\psi_n\right)\right|_{z=h}=0 \tag{16}$$

The functions $\psi_n(z)$ are expressed as [7]

$$\psi_n(z)=\beta_n\cos\beta_n z+\mu_0\sin\beta_n z,\quad n=1,2,... \tag{17}$$

These functions are pairwise orthogonal, so that the following condition is satisfied

$$\int_0^h\psi_n(z)\psi_m(z)dz=\begin{cases}0 & \text{for}\, n\neq m\\ Q_n & \text{for}\, n=m\end{cases} \tag{18}$$

where

$$Q_n=\int_0^h(\psi_n(z))^2dz=\frac{h}{2}\left(\beta_n^2+\mu_0^2\right)\left(1+\frac{\beta_n^2+\mu_0{}^2}{2\mu_0 h\beta_n^2}\sin^2\beta_n h\right) \tag{19}$$

and β_n are roots of the equation

$$2\mu_0\,\beta_n\cos\beta_n h-\left(\beta_n^2-\mu_0{}^2\right)\sin\beta_n h=0 \tag{20}$$

The Dirac function $\delta(z-\zeta)$ in equation (11) may be written in the form

$$\delta(z-\zeta)=\sum_{n=1}^{\infty}\frac{\psi_n(z)\psi_n(\zeta)}{Q_n} \tag{21}$$

Substituting (14) and (21) into equation (11), we obtain the differential equation

$$\frac{\partial^2 \Gamma_{mn}}{\partial r^2}+\frac{1}{r}\frac{\partial \Gamma_{mn}}{\partial r}-\left(\beta_n^2+\frac{m^2}{r^2}\right)\Gamma_{mn}-\frac{1}{\kappa}\frac{\partial \Gamma_{mn}}{\partial t}=\frac{\psi_n(\zeta)}{Q_n}\frac{\delta(r-\rho)\delta(t-\tau)}{2\pi r} \tag{22}$$

The initial and boundary conditions are given as

$$\begin{gathered}\Gamma_{mn}(r,0)=0\\ \Gamma_{mn}\big|_{r=a}=0 \qquad \Gamma_{mn}\big|_{r=b}=0\\ \left(\frac{\partial \Gamma_{mn}}{\partial r}-\mu_0\Gamma_{mn}\right)\bigg|_{r=a}=0 \qquad \left(\frac{\partial \Gamma_{mn}}{\partial r}-\mu_0\Gamma_{mn}\right)\bigg|_{r=b}=0\end{gathered} \tag{23}$$

In order to solve the problem (22)-(23), the function $\Gamma_{mn}(r,t)$ can be written in the form

$$\Gamma_{mn}(r,t)=\sum_{k=1}^{\infty}R_{mnk}(r)T_{mnk}(t), \quad a\le r\le b \tag{24}$$

where functions $R_{mnk}(r)$ satisfy the Bessel's equation

$$\frac{\partial^2 R_{mnk}}{\partial r^2}+\frac{1}{r}\frac{\partial R_{mnk}}{\partial r}+\left(\gamma_{mnk}^2-\frac{m^2}{r^2}\right)R_{mnk}(r)=0 \tag{25}$$

where γ_{mnk} are constants. The functions R_{mnk} satisfy the following conditions

$$R_{mnk}(a)=0, \quad R_{mnk}(b)=0 \tag{26}$$

The general solution of equation (25) is expressed by well-known Bessel functions in the form

$$R_{mnk}(r)=C_1J_m(\gamma_{mnk}r)+C_2Y_m(\gamma_{mnk}r) \tag{27}$$

where J_m, Y_m are the Bessel functions. Using (27) in the boundary conditions (26), we obtain a system of two homogeneous equations with unknowns C_1, C_2.

$$\begin{cases}C_1J_m(\gamma_{mnk}a)+C_2Y_m(\gamma_{mnk}a)=0\\ C_1J_m(\gamma_{mnk}b)+C_2Y_m(\gamma_{mnk}b)=0\end{cases} \tag{28}$$

The non-trivial solution of the system (28) exists if γ_{mnk} are roots of the following characteristic equation

$$J_m(\gamma_{mnk}a)Y_m(\gamma_{mnk}b)-J_m(\gamma_{mnk}b)Y_m(\gamma_{mnk}a)=0 \tag{29}$$

The unknowns C_2 may be found from the first equation of the system (28) as

$$C_2 = -\frac{J_m(\gamma_{mnk} a)}{Y_m(\gamma_{mnk} a)} C_1 \tag{30}$$

The functions R_{mnk} may be assumed in the form

$$R_{mnk}(r) = Y_m(\gamma_{mnk} a) J_m(\gamma_{mnk} r) - J_m(\gamma_{mnk} a) Y_m(\gamma_{mnk} r) \tag{31}$$

Note that the functions R_{mnk} satisfy the orthogonality condition

$$\int_0^b r R_{mnk}(r) R_{mnk'}(r) dr = \begin{cases} 0 & \text{for } k' \neq k \\ \chi_{mnk} & \text{for } k' = k \end{cases} \tag{32}$$

where:

$$\chi_{mnk} = \phi_{mnk}^{[1]} J_m^2(a\gamma_{mnk}) + \phi_{mnk}^{[2]} Y_m^2(a\gamma_{mnk}) - 2\phi_{mnk}^{[3]} JI_m(a\gamma_{mnk}) Y_m(a\gamma_{mnk}) \tag{33}$$

$$\phi_{mnk}^{[1]} = -\frac{a^2}{2}\left(Y_{m-1}^2(a\gamma_{mnk}) + Y_m^2(a\gamma_{mnk})\right) + \frac{b^2}{2}\left(Y_{m-1}^2(b\gamma_{mnk}) + Y_m^2(b\gamma_{mnk})\right) + \frac{m}{\gamma_{mnk}}\left(a Y_{m-1}(a\gamma_{mnk}) Y_m(a\gamma_{mnk}) - b Y_{m-1}(b\gamma_{mnk}) Y_m(b\gamma_{mnk})\right)$$

$$\phi_{mnk}^{[2]} = -\frac{a^2}{2}\left(J_{m-1}^2(a\gamma_{mnk}) + J_m^2(a\gamma_{mnk})\right) + \frac{b^2}{2}\left(J_{m-1}^2(b\gamma_{mnk}) + J_m^2(b\gamma_{mnk})\right) + \frac{m}{\gamma_{mnk}}\left(a J_{m-1}(a\gamma_{mnk}) J_m(a\gamma_{mnk}) - b J_{m-1}(b\gamma_{mnk}) J_m(b\gamma_{mnk})\right)$$

$$\phi_{mnk}^{[3]} = \frac{1}{2\sqrt{\pi}}\left[b^2 G_{3,5}^{2,2}\left((b\gamma_{mnk})^2 \middle| \begin{matrix} 0, 0.5, -0.5 \\ 0, m, -m, -1, -0.5 \end{matrix}\right) - a^2 G_{2,4}^{2,2}\left((a\gamma_{mnk})^2 \middle| \begin{matrix} 0, 0.5, -0.5 \\ 0, m, -m, -1, -0.5 \end{matrix}\right)\right]$$

where $G_{p,q}^{m,n}\left(x \middle| \begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix}\right)$ is a G-Meijer function [8].

Substituting the function Γ_{mnk} defined by (24) into (22), multiplying by $R_{mnk'}$, then integrating both sides of the obtained equation in the interval (a,b) and using the orthogonality condition (32), we have

$$\frac{dT_{mnk}}{dt} + \kappa\left(\gamma_{mnk}^2 + \beta_n^2\right) T_{mnk} = -\frac{\kappa \psi_n(\zeta) R_{mnk}(\rho)}{2\pi Q_n \chi_{mnk}} \delta(t - \tau) \tag{34}$$

The solution of this equation with zero initial condition is as follows

$$T_{mnk}(t) = -\frac{\kappa}{2\pi}\frac{\psi_n(\zeta)R_{mnk}(\rho)}{Q_n\,\chi_{mnk}}e^{-\kappa\left(\gamma_{mnk}^2+\beta_n^2\right)(t-\tau)} \tag{35}$$

Finally, the Green's function on the basis of equations (9), (14), (24) and (35) assumes the following form

$$G(r,\phi,z,t,\rho,\phi',\varsigma,\tau) = -\frac{\kappa}{2\pi}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\psi_n(z)\psi_n(\zeta)}{Q_n\chi_{mnk}}R_{mnk}(r)R_{mnk}(\rho)e^{-\kappa\left(\gamma_{mnk}^2+\beta_n^2\right)(t-\tau)}\cos m(\phi-\phi') \tag{36}$$

where the functions ψ_n are defined by equation (17), R_{mnk} are given by equation (31), Q_n, are given by (19), χ_{mnk} are given by (33), β_n are roots of equation (20) and γ_{mnk} are roots of equation (29).

The temperature distribution $T(r,\phi,z,t)$ is expressed by the Green's function G as follows

$$T(r,\phi,z,t) = \int_0^t d\tau\int_0^b d\rho\int_0^{2\pi} d\phi\int_0^h g(\rho,\phi',\varsigma,\tau)G(r,\phi,z,t;\rho,\phi',\varsigma,\tau)dz \tag{37}$$

After evaluation of the integrals respect to ρ, ϕ, z and use equation (2) we obtain the temperature $T(r,\phi,z,t)$ in the form

$$T(r,\phi,z,t) = \theta\int_0^t G(r,\phi,z,t;r_0,\varphi(\tau\},h,\tau)d\tau \tag{38}$$

Substitution of the Green's function (36) into equation (38) gives

$$T(r,\phi,z,t) = \frac{\theta\kappa}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{R_{mnk}(\rho)\psi_n(h)}{Q_n\,\chi_{mnk}}R_{mnk}(r)\Psi_n(z)P_{mnk}(t,\theta) \tag{39}$$

where

$$P_{mnk}(t,\phi) = \int_0^t \cos m(\phi-\varphi(\tau))\exp\left[-\kappa\left(\beta_n^2+\gamma_{mnk}^2\right)(t-\tau)\right]d\tau \tag{40}$$

After evaluation of the integral we have

$$P_{mnk}(t,\phi)=\frac{1}{\vartheta_{mnk}^2+m^2\omega^2}\Big[\vartheta_{mnk}\cos m(\phi-\omega t)-m\omega\sin m(\phi-\omega t) \\ -\left(\vartheta_{mnk}\cos m\phi-m\omega\sin m\phi\right)e^{-\vartheta_{mnk}t}\Big] \tag{41}$$

where $\vartheta_{mnk}=\kappa\left(\beta_n^2+\gamma_{mnk}^2\right)$.

Conclusions

In this paper the temperature distribution in an annular plate with point heat source was considered. It was assumed that the heat source moves on plate surface along a concentric circular trajectory. The solution of the problem by using the time-dependent Green's function in an analytical form has been obtained. Finally, the three-dimensional temperature field for the plate in a series form was presented. The obtained solution can be used in the numerical analysis of the field temperature of the plate as well as for an investigation of the thermally induced vibration of this plate.

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