APPLICATION OF THE BOUNDARY ELEMENT METHOD COUPLED WITH THE ARTIFICIAL HEAT SOURCE PROCEDURE FOR NUMERICAL MODELLING OF FREEZING PROCESS

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Abstract. The freezing process of biological tissue subjected to the action of external cylindrical cryoprobe is analyzed. From the mathematical point of view the problem belongs to the group of moving boundary ones because the shape and dimensions of frozen region are time-dependent. In the paper the mathematical model of the process considered and the method of solution are presented. Next, the example of computations is shown.

1. Mathematical model

The freezing process proceeding in the domain of biological tissue subjected to the action of external cylindrical cryoprobe (Fig. 1) can be described by following equation [1-6]

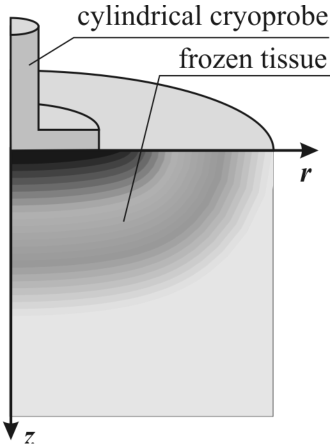

Fig. 1. Domain considered

$$(r,z)\in\Omega:\quad C(T)\frac{\partial T(r,z,t)}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[r\lambda(T)\frac{\partial T(r,z,t)}{\partial r}\right]+\frac{\partial}{\partial z}\left[\lambda(T)\frac{\partial T(r,z,t)}{\partial z}\right]\qquad(1)$$

where $C(T)$ is the substitute thermal capacity, $\lambda(T)$ is the thermal conductivity, T, $\{r, z\}$, t denote temperature, cylindrical co-ordinate system and time.

The functions corresponding to substitute thermal capacity $C(T)$ and thermal conductivity $\lambda(T)$ [7, 8] are shown in Figures 2 and 3, respectively. It should be pointed out that natural state of biological tissue corresponds to the temperatures $T > T_1 = -1$°C, while frozen state corresponds to the temperatures $T < T_2 = -8$°C. In the interval of temperature $[T_2, T_1]$ the evolution of latent heat is taken into account.

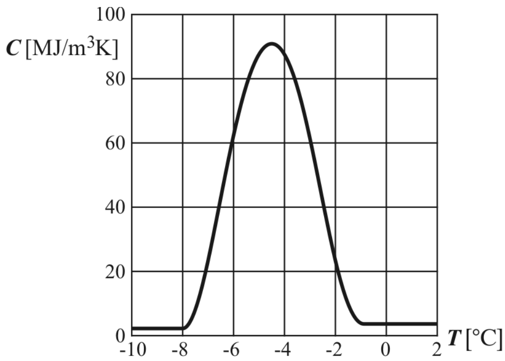

Fig. 2. Function $C(T)$

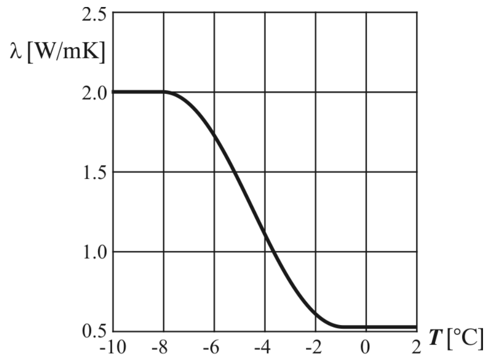

Fig. 3. Function $\lambda(T)$

On the contact surface Γ_c between cryoprobe and tissue the Dirichlet boundary condition is assumed

$$(r,z)\in\Gamma_c: \quad T_b(t)=T_0-w_z t, \quad t\le t_b \tag{2}$$

where w_z [K/s] is the cooling rate, T_0 is the initial temperature of cryoprobe and t_b is the freezing time. After the time t_b the boundary condition (2) is changed, namely

$$(r,z)\in\Gamma_c: \quad T_b(t)=T_b(t_b)-w_r t, \quad t>t_b \tag{3}$$

where w_r [K/s] is the heating rate, while $T_b(t_b)$ is the minimum temperature of cryoprobe tip.

On the remaining boundary Γ_0 limiting the domain considered the no-flux condition can be accepted

$$(r,z)\in\Gamma_0: \quad q(r,z,t)=-\lambda(T)\,\mathbf{n}\cdot\nabla T(r,z,t)=0 \tag{4}$$

Mathematical model is supplemented by initial condition

$$t=0: \quad T(r,z,0)=T_p \tag{5}$$

where T_p is the initial temperature of tissue.

It should be pointed out that the problem discussed is strongly non-linear and in order to construct the effective algorithm basing on the boundary element method the additional numerical procedure 'linearizing' the mathematical model (on a stage of computations) should be taken into account. Here the artificial heat source method [9] is applied.

2. Kirchhoff's transformation

To linearize the mathematical model of freezing process the Kirchhoff substitution is introduced

$$U(T)=\int_{T_r}^{T}\lambda(\mu)\,\mathrm{d}\,\mu \tag{6}$$

where T_r is the reference level (in computations it is assumed that $T_r = -200°C$). Because

$$\frac{\partial U(r,z,t)}{\partial r}=\frac{\mathrm{d}U}{\mathrm{d}T}\frac{\partial T(r,z,t)}{\partial r}=\lambda(T)\frac{\partial T(r,z,t)}{\partial r} \tag{7}$$

and

$$\frac{\partial U(r,z,t)}{\partial z}=\frac{\mathrm{d}U}{\mathrm{d}T}\frac{\partial T(r,z,t)}{\partial z}=\lambda(T)\frac{\partial T(r,z,t)}{\partial z} \tag{8}$$

while

$$\frac{\partial U(r,z,t)}{\partial t}=\frac{\mathrm{d}U}{\mathrm{d}T}\frac{\partial T(r,z,t)}{\partial t}=\lambda(T)\frac{\partial T(r,z,t)}{\partial t} \tag{9}$$

so the right-hand side of equation (1) takes a form

$$\begin{gathered}\frac{1}{r}\frac{\partial}{\partial r}\left[\lambda(T)\,r\frac{\partial T(r,z,t)}{\partial r}\right]+\frac{\partial}{\partial z}\left[\lambda(T)\frac{\partial T(r,z,t)}{\partial z}\right]=\\ \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial U(r,z,t)}{\partial r}\right]+\frac{\partial^2 T(r,z,t)}{\partial z^2}\end{gathered} \tag{10}$$

and left-hand side of equation (1) can be written as

$$C(T)\frac{\partial T(r,z,t)}{\partial t}=\frac{C(T)}{\lambda(T)}\frac{\partial U(r,z,t)}{\partial t} \tag{11}$$

Finally, using the Kirchhoff convention one obtains

$$\Phi[T(U)]\frac{\partial U(r,z,t)}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial U(r,z,t)}{\partial r}\right]+\frac{\partial^2 T(r,z,t)}{\partial z^2} \tag{12}$$

where

$$\Phi[T(U)]=\frac{C[T(U)]}{\lambda[T(U)]} \tag{13}$$

In Figures 4 and 5 the courses of functions $U(T)$ and $\Phi[T(U)]$ are shown.

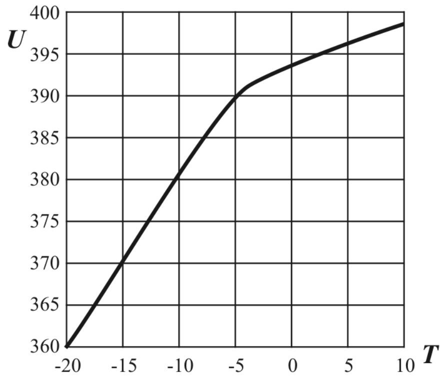

Fig. 4. Function $U(T)$

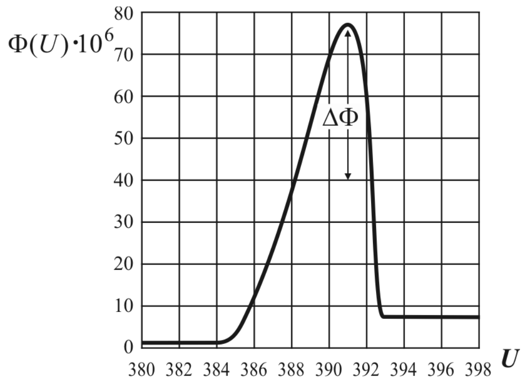

Fig. 5. Function $\Phi[T(U)]$

The boundary and initial conditions should be also written using Kirchhoff's transformation, this means

$$
\begin{aligned}
&(r,z)\in\Gamma_c: && U(r,z,t)=U_b(t)=U[T_b(t)]\\
&(r,z)\in\Gamma_0: && q(r,z,t)=-\mathbf{n}\cdot\nabla U(r,z,t)=0\\
&t=0: && U(r,z,0)=U(T_p)=U_p
\end{aligned}
\tag{14}
$$

3. Artificial heat source method

Function $\Phi[T(U)]$ (c.f. equation (12)) can be expressed as a sum of two components

$$
\Phi[T(U)]=\Phi_0+\Delta\Phi[T(U)] \tag{15}
$$

The first component is constant, while the second denotes a certain increment as shown in Figure 5.

The equation (12) can be written in the form

$$
\begin{aligned}
\Phi_0\frac{\partial U(r,z,t)}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial U(r,z,t)}{\partial r}\right]\\
&+\frac{\partial^2 T(r,z,t)}{\partial z^2}-\Delta\Phi[T(U)]\frac{\partial U(r,z,t)}{\partial t}
\end{aligned}
\tag{16}
$$

or

$$\Phi_0 \frac{\partial U(r,z,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[r \frac{\partial U(r,z,t)}{\partial r}\right] + \frac{\partial^2 T(r,z,t)}{\partial z^2} + Q_a(r,z,t) \tag{17}$$

where

$$Q_a(r,z,t) = -\Delta\Phi[T(U)]\frac{\partial U(r,z,t)}{\partial t} \tag{18}$$

is the artificial heat source term. The calculation of a source function requires, of course, the introduction of a certain iterative procedure [9, 10]. It should be pointed out that if $|\Delta\Phi[T(U)]| < \Phi_0$ then the adequate iterative algorithm is convergent.

4. 1st scheme of the boundary element method

The essential feature of equation (17) consists in a fact, that leaving out the last term we obtain the linear form of energy equation. Taking into account the possibilities of the boundary element method application in the range of non-steady problems modelling, this is a very convenient form of basic differential equation because a non-linearity appears only in the component determining the internal heat sources.

It should be pointed out that in order to use the BEM for domains oriented in Cartesian co-ordinate system the equation (17) can be expressed as follows [11]

$$\Phi_0 \frac{\partial U(x,y,z)}{\partial t} = \frac{\partial^2 U(x,y,t)}{\partial x^2} + \frac{\partial^2 U(x,y,t)}{\partial y^2} + S(x,y,t) \tag{19}$$

where

$$S(x,y,t) = \frac{1}{x}\frac{\partial U(x,y,t)}{\partial x} + Q_a(x,y,t) \tag{20}$$

is a new artificial source function and $x = r$, $y = z$.

Application of the 1st scheme of the BEM to equation (19) for transition $t^{f-1} \to t^f$ and constant elements with respect to time and spatial co-ordinates leads to the following system of equations [12, 13]

$$\sum_{j=1}^{N} G_{ij} q_j^f = \sum_{j=1}^{N} H_{ij} U_j^f + \sum_{l=1}^{L} P_{il} U_l^{f-1} + \sum_{l=1}^{L} Z_{il} S_l^f \tag{21}$$

where G_{ij}, H_{ij}, P_{il}, Z_{il} are elements of influence matrices [13].

In the first step of computations the system of equations (21) is solved and next the internal values of function U are calculated using the formula

$$U_i^f = \sum_{j=1}^{N} H_{ij} U_j^f - \sum_{j=1}^{N} G_{ij} q_j^f + \sum_{l=1}^{L} P_{il} U_l^{f-1} + \sum_{l=1}^{L} Z_{il} S_l^f \tag{22}$$

5. Results of computations

The radius of cryoprobe equals $R_c = 0.005$ m and the tissue domain of dimensions $3R_c \times 3R_c$ has been considered. The boundary has been divided into 60 constant boundary elements and the interior has been divided into 225 internal cells (squares) as shown in Figure 6. Time step: $\Delta t = 10$ s. It was assumed that cooling rate is equal to heating rate $w_z = w_r = 5$ K/min and freezing time equals $t_b = 900$ s. In Figure 7 the temperature distribution for times 5, 10, 15 and 20 minutes is shown.

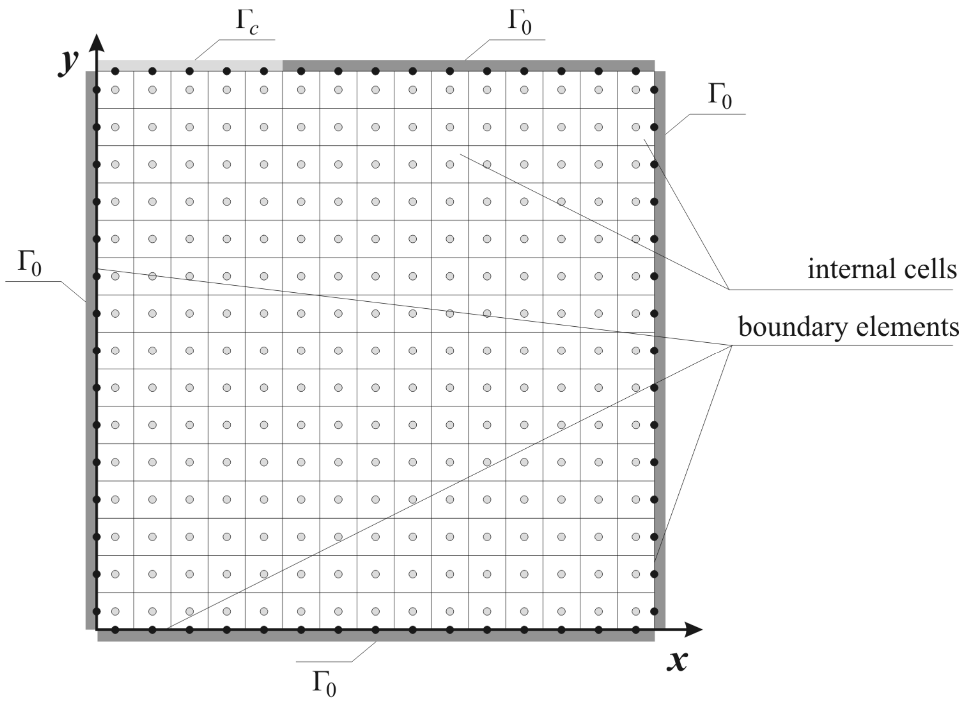

Fig. 6. Discretization

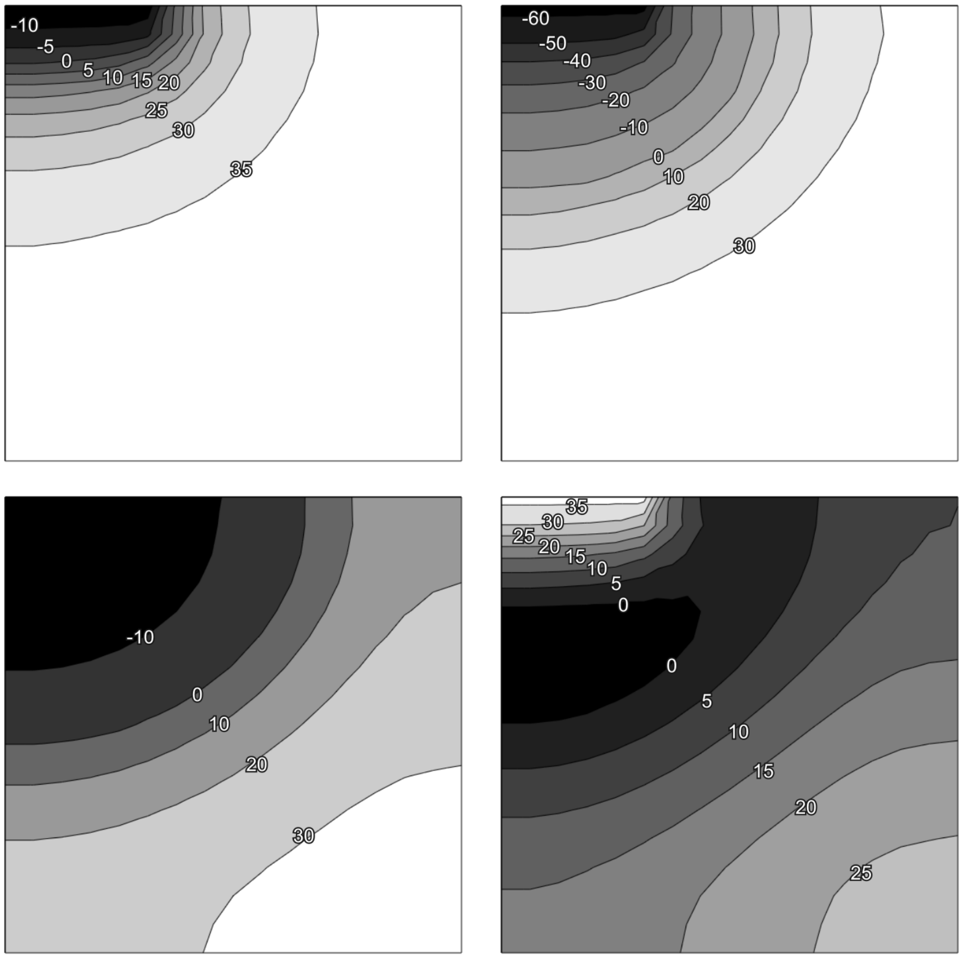

Fig 7. Temperature distribution for 5, 10, 15 and 20 minutes

Conclusions

The 1st scheme of the boundary element method coupled with the artificial heat source procedure has been applied for numerical modelling of biological tissue freezing process. The results of computations showed that the algorithm proposed constitutes an effective tool of such problems solutions.

Acknowledgement

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