ANALYSIS OF HM-NETWORKS WITH STOCHASTIC INCOMES FROM TRANSITIONS BETWEEN STATES

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Abstract. In article method of finding of expected incomes in systems of HM-network of arbitrary topology when incomes from transitions between network states are stochastic variables with given mean values is proposed. For expected incomes the system of linear non-homogeneous ordinary differential equations was obtained, to solve it we can find incomes in network systems.

Introduction

Markov queueing networks (QN) with incomes were examined in works [1-3] at the first time. They are described with help of Markov chains with continuous time and incomes which were introduced by R. Howard [4]. So at the recent time they are called HM(Howard-Matalytski)-networks [5, 6]. Before investigation of closed networks with account number of states was carry out. Herewith the next cases were examined: a) incomes from transitions between network states depend on states and time or b) incomes are stochastic variables (SV) with known finite moments of the first and the second orders.

For expected incomes of network systems in case a) the system of difference-differential equations which are reduced to the system of linear non-homogeneous ordinary differential equations (ODE) can be obtained. For their solution different methods - method of multidimensional z-transformations and known methods: method of Laplace transformation, matrix method, numerical methods were proposed [7-9].

In [5, 10] approximate relations for expected incomes and income variations in systems of exponential HM-networks in case b) were obtained. Technique of receiving of these relations is based on interval partition of the network functioning by big number m of small intervals of size Δt, income estimation on every interval and summing of these incomes by means of passage to the limit $m \to \infty$, $\Delta t \to 0$. Herewith mean value of messages in network system in unsteady condition was found with help of developed recurrence by time moments method. Notice that different methods of analysis and optimization of Markov HM-networks and their application were described in [11]. In the present work method of finding of expected incomes of network systems in case b) which is based on solution of the linear non-homogeneous ODE, which were received for expected incomes and mean values of messages, is proposed.

1. Finding of expected incomes in systems

Let us examine open exponential QN of arbitrary topology with one type messages which consists of n queueing systems (QS) $S_1, S_2,\dots,S_n$ with m_i service channels in system S_i, $i=\overline{1,n}$. Denote via $k(t)=(k_1(t),k_2(t),...,k_n(t))$ - vector of network states, where $k_i(t)$ - message number in system S_i (in queue and service) at the moment t. Poisson flow of messages of the rate λ enters the network. Service rate of messages at the moment t $\mu_i(k_i(t))$ in system S_i depends on message number in this system, $i=\overline{1,n}$. Message when transiting from one QS to another brings to the last system some stochastic income and income of the first system reduces by this value correspondingly.

Let us consider dynamics of income changes of some network system S_i. Denote it's income at the moment t as $V_i(t)$. Let at the initial moment income of system equals $V_i(0) = v_{i0}$. Income of this QS at the moment $t + \Delta t$ can be presented as

$$V_i\ (t+\Delta t)=V_i(t)+\Delta V_i(t,\Delta t) \tag{1}$$

where $\Delta V_i(t,\Delta t)$ - income change of system S_i on time interval $[t, t+\Delta t)$. For finding of this value we write probabilities of events which can appear during time Δt and changes of incomes of system S_i which are connected with these events.

1. Message from the outside with probability $\lambda p_{0i}\Delta t + o(\Delta t)$ will enter to system S_i and will bring to it income of size r_{0i}, where r_{0i} - RV with mathematical expectation (m.e.) $M\{r_{0i}\}=a_{0i}$, p_{0i} - probability of message enter from outside to the system S_i, $i=\overline{1,n}$.
2. Message from the system S_i with probability $\mu_i(k_i(t))u(k_i(t))p_{i0}\Delta t+o(\Delta t)$ will pass to the outside and income of the system S_i will decrease by value R_{i0}, where R_{i0} - RV with m.e. $M\{R_{i0}\}=b_{i0}$, p_{i0} - probability of message leaving from system S_i to the outside, $i=\overline{1,n}$, $u(x)=\begin{cases}1, & x>0,\\ 0, & x\le 0,\end{cases}$ - Heavyside function.
3. Message from the system S_j with probability $\mu_j(k_j(t))u(k_j(t))p_{ji}\Delta t+o(\Delta t)$ will pass to the system S_i and income of the system S_i will increase by value r_{ji} and income of the system S_j will decrease by this value, where r_{ji} - RV with m.e.

$M\left\{r_{ji}\right\}=a_{ji}$, p_{ji} - probability of message transition from the system S_j to the system S_i, $i,j=\overline{1,n}$, $i\neq j$.

4. Message from the system S_i with probability $\mu_i(k_i(t))u(k_i(t))p_{ij}\Delta t+o(\Delta t)$ will pass to the system S_j and income of the QS S_i will decrease by value R_{ij} and income of the system S_j increase by this value, where R_{ij} - RV with m.e. $M\left\{R_{ij}\right\}=b_{ij}$, $i,j=\overline{1,n}$, $i\neq j$.
5. State changes of system S_i on the time interval $[t,t+\Delta t)$ with probability $1-\left(\lambda p_{0i}+\mu_i(k_i(t))u(k_i(t))+\sum\limits_{\substack{j=1\\ j\neq i}}^{n}\mu_j(k_j(t))u(k_j(t))p_{ji}\right)\Delta t+o(\Delta t)$ will not appear, $i=\overline{1,n}$.

Besides that during each small time interval Δt the system S_i increases its income by value $r_i\,\Delta t$, where r_i - RV with m.e. $M\{r_i\}=c_i$, $i=\overline{1,n}$. Let also suppose that RV r_{ji}, R_{ij}, r_{0i}, R_{i0} are independent with respect to RV r_i, $i,j=\overline{1,n}$.

Evidently that $r_{ji}=R_{ji}$ with probability 1, i.e.

$$a_{ji}=b_{ji},\ \ i,j=\overline{1,n} \tag{2}$$

Then from said follows

$$\Delta V_i\ (t,\Delta t)=\begin{cases} r_{0i}+r_i\,\Delta t & \text{with probability} \quad \lambda p_{0i}\Delta t+o(\Delta t),\\ -R_{i0}+r_i\,\Delta t & \text{with probability} \quad \mu_i(k_i(t))u(k_i(t))p_{i0}\Delta t+o(\Delta t),\\ r_{ji}+r_i\,\Delta t & \text{with probability} \quad \mu_j(k_j(t))u(k_j(t))p_{ji}\Delta t+o(\Delta t),\\ -R_{ij}+r_i\,\Delta t & \text{with probability} \quad \mu_i(k_i(t))u(k_i(t))p_{ij}\Delta t+o(\Delta t),\\ r_i\,\Delta t & \text{with probability} \quad 1-\Big(\lambda p_{0i}+\mu_i(k_i(t))u(k_i(t))+\\ & \qquad +\sum\limits_{\substack{j=1\\ j\neq i}}^{n}\mu_j(k_j(t))u(k_j(t))p_{ji}\Big)\Delta t+o(\Delta t) \end{cases} \tag{3}$$

Under fixed realization of the process $k(t)$ and taking to account (3) it can be written:

$$M\{\Delta V_i(t,\Delta t)/k(t)\}=\left[\lambda p_{0i}a_{0i}+c_i-\mu_i(k_i(t))u(k_i(t))\left(p_{i0}b_{i0}+\sum_{\substack{j=1\\ j\neq i}}^{n}p_{ij}b_{ij}\right)+\right.$$

$$\left.+\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_j(k_j(t))u(k_j(t))p_{ji}a_{ji}\right]\Delta t+o(\Delta t)$$

Average the last relation by $k(t)$ with taking to account normalization condition $\sum_k P(k(t)=k)=1$ for income change of system S_i we will obtain

$$M\{\Delta V_i(t,\Delta t)\}=\sum_k P(k(t)=k)M\{\Delta V_i(t,\Delta t)/k(t)\}=$$

$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k(t)=(k_1(t),k_2(t),...,k_n(t)))\times$$

$$\times M\{\Delta V_i(t,\Delta t)/k(t)=(k_1(t),k_2(t),...,k_n(t))\}=$$

$$=\left[\lambda p_{0i}a_{0i}+c_i-\left(p_{i0}b_{i0}+\sum_{\substack{j=1\\ j\neq i}}^{n}p_{ij}b_{ij}\right)\sum_k P(k(t)=k)\mu_i(k_i(t))u(k_i(t))+\right.$$

$$\left.+\sum_{\substack{j=1\\ j\neq i}}^{n}p_{ji}a_{ji}\sum_k P(k(t)=k)\mu_j(k_j(t))u(k_j(t))\right]\Delta t+o(\Delta t)$$

Let the system S_i contains m_i identical service channels, time of message service in every channel is distributed under exponential law with parameter μ_i, $i=\overline{1,n}$. In this case

$$\mu_i(k_i(t))=\begin{cases}\mu_i k_i(t), & k_i(t)\le m_i,\\ \mu_i m_i, & k_i(t)>m_i,\end{cases}\quad \mu_i(k_i(t))u(k_i(t))=\mu_i\min(k_i(t),m_i),\quad i=\overline{1,n}$$

Let us suppose that averaging of expression $\mu_i(k_i(t))u(k_i(t))$ brings $\mu_i\min(N_i(t),m_i)$, i.e.

$$M\min(k_i(t),m_i)=\min(N_i(t),m_i) \tag{4}$$

where $N_i(t)$ - average number of messages (waiting and serving) in the system S_i at the moment t, $i=\overline{1,n}$. Taking in account this assumption we obtain the next approximate relation

$$M\{\Delta V_i(t,\Delta t)\}=\left[\lambda p_{0i}a_{0i}+c_i-\mu_i\min(N_i(t),m_i)\left(p_{i0}b_{i0}+\sum_{\substack{j=1\\j\neq i}}^{n}p_{ij}b_{ij}\right)+\right.$$
$$\left.+\sum_{\substack{j=1\\j\neq i}}^{n}\mu_j\min(N_j(t),m_j)p_{ji}a_{ji}\right]\Delta t+o(\Delta t) \quad (5)$$

Since Poisson flow of messages with rate λ enters the network, i.e. probability of entering of l messages in the system S_i during time Δt has the appearance $P_l(\Delta t)=\frac{(\lambda p_{0i}\Delta t)^l}{l!}e^{-\lambda p_{0i}\Delta t}$, $l=0,1,2,...$, so average number of messages which entered to the system S_i from the outside during time Δt equals $\lambda p_{0i}\Delta t$. Denote the average number of busy service channels in system S_i at the moment t as $\rho_i(t)$, $i=\overline{1,n}$. Then $\mu_i\rho_i(t)\Delta t$ - average number of messages that left the system S_i during time Δt and $\sum_{\substack{j=1\\j\neq i}}^{n}\mu_j\rho_j(t)p_{ji}\Delta t$ - average number of messages that entered in the system S_i from another QS during time Δt. So

$$N_i(t+\Delta t)-N_i(t)=\lambda p_{0i}\Delta t+\sum_{\substack{j=1\\j\neq i}}^{n}\mu_j\rho_j(t)p_{ji}\Delta t-\mu_i\rho_i(t)\Delta t,\ i=\overline{1,n}$$

whence with $\Delta t\to 0$ system of ODE for $N_i(t)$ follows:

$$\frac{dN_i(t)}{dt}=\sum_{\substack{j=1\\j\neq i}}^{n}\mu_j\rho_j(t)p_{ji}-\mu_i\rho_i(t)+\lambda p_{0i},\ i=\overline{1,n} \quad (6)$$

It is impossible to find variable $\rho_i(t)$ exactly so as we did earlier we approximate it by expression

$$\rho_i(t)=\begin{cases}N_i(t),\ N_i(t)\le m_i,\\ m_i, N_i(t)>m_i,\end{cases}=\min(N_i(t),\ m_i)$$

Then system of equations (6) will take on form

$$\frac{dN_i(t)}{dt} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \mu_j p_{ji} \min(N_j(t), m_j) - \mu_i \min(N_i(t), m_i) + \lambda p_{0i}, \ i = \overline{1,n} \tag{7}$$

This system is system of linear ODE with discontinuous right parts. It's necessary to solve it by means of segmentation of the phase space by set of the areas and finding solution in each of them. System (7) can be solved for example by using means of system of computer mathematics Maple 8.

Let us introduce denotation $v_i(t) = M\{V_i(t)\}$, $i = \overline{1,n}$. From (1), (5) we obtain

$$v_i\,(t + \Delta t) = v_i(t) + M\{\Delta V_i(t, \Delta t)\} =$$

$$= v_i(t) + \left[\lambda p_{0i} a_{0i} + c_i \ - \mu_i \min(N_i(t),\ m_i) \left(p_{i0} b_{i0} + \sum_{\substack{j=1 \\ j \neq i}}^{n} p_{ij} b_{ij} \right) + \right.$$

$$\left. + \sum_{j=1}^{n} \mu_j \min(N_j(t),\ m_j) p_{ji} a_{ji} \right] \Delta t + o(\Delta t)$$

Then pass to limit with $\Delta t \to 0$ we receive non-homogeneous linear ODE of the first order

$$\begin{aligned} \frac{dv_i(t)}{dt} &= -\mu_i \min(N_i(t), m_i) \left(p_{i0} b_{i0} + \sum_{\substack{j=1 \\ j \neq i}}^{n} p_{ij} b_{ij} \right) + \\ &+ \sum_{\substack{j=1 \\ j \neq i}}^{n} \mu_j \min(N_j(t), m_j) p_{ji} a_{ji} + \lambda p_{0i} a_{0i} + c_i, \ i = \overline{1,n} \end{aligned} \tag{8}$$

Specify initial conditions $v_i(0) = v_{i0}$, $i = \overline{1,n}$, it is possible to find expected incomes of network systems.

If the network is functioning so that queues are not observed in it upon the average, i.e. $\min(N_i(t), m_i) = N_i(t)$, $i = \overline{1,n}$, then systems (7), (8) will take the appearance:

$$\frac{dN_i(t)}{dt} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \mu_j p_{ji} N_j(t) - \mu_i N_i(t) + \lambda p_{0i}, \ i = \overline{1,n} \tag{9}$$

$$\begin{cases} \dfrac{dv_i(t)}{dt} = -\mu_i \left(p_{i0}b_{i0} + \sum\limits_{j=1}^{n} p_{ij}b_{ij} \right) N_i(t) + \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \mu_j p_{ji} a_{ji} N_j(t) + \lambda p_{0i} a_{0i} + c_i \,, \\ v_i(0) = v_{i0}, \ i = \overline{1,n} \end{cases} \quad (10)$$

System (9) can be rewrote in matrix form

$$\frac{dN(t)}{dt} = QN(t) + f \quad (11)$$

where $N^T(t) = (N_1(t), N_2(t), ..., N_n(t))$, Q - quadratic matrix which consists of elements $q_{ij} = \mu_j p_{ji}$, if suppose $p_{ii} = -1$, $i, j = \overline{1,n}$, f - column vector with elements λp_{0i}, $i = \overline{1,n}$. Solution of the system (11) is

$$N(t) = N(0)e^{Qt} + f \int_0^t e^{Q(t-\tau)} d\tau$$

where $N(0)$ - some given initial conditions, but finding of elements of matrix e^{Qt} is difficult problem even for rather small values of n.

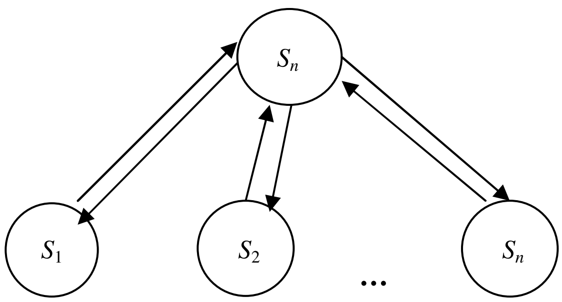

Fig. 1. Structure of the network with central QS

Let us consider closed network with central QS that consists of n systems (Fig. 1). Let suppose that queues are not observed in peripheral systems of the network under the average, i.e. $\min(N_i(t), m_i) = N_i(t)$, $i = \overline{1, n-1}$, and central QS functions in the condition of heavy traffic, i.e. $\min(N_n(t), m_n) = m_n$. System (7) in this case will rewrite as

$$\begin{cases} \dfrac{dN_i(t)}{dt} = -\mu_i N_i(t) + \mu_n m_n p_{ni}, \ i = \overline{1, n-1}, \\ \dfrac{dN_n(t)}{dt} = \sum\limits_{i=1}^{n-1} \mu_i N_i(t) - \mu_n m_n \end{cases} \quad (12)$$

General solution of the system (12) with initial conditions $N_i(0),\ i=\overline{1,\ n}$, equals

$$N_i(t)=g_i e^{-\mu_i t}+\frac{m_n\mu_n p_{ni}}{\mu_i},\ i=\overline{1,n-1},$$
$$N_n(t)=K-\sum_{i=1}^{n-1}\left(g_i e^{-\mu_i t}+\frac{m_n\mu_n p_{ni}}{\mu_i}\right)$$

where $g_i=N_i(0)-\frac{m_n\mu_n p_{ni}}{\mu_i}$, $K=\sum_{i=1}^{n}N_i(t)$ - number of messages in the network. For such $N_i(t),\ i=\overline{1,\ n}$, system (10) for expected incomes of the network systems will take the appearance

$$\frac{dv_i(t)}{dt}=-\mu_i b_{in}\left(g_i e^{-\mu_i t}+\frac{m_n\mu_n p_{ni}}{\mu_i}\right)+$$
$$+\mu_n p_{ni}a_{ni}\left[K-\sum_{j=1}^{n-1}\left(g_j e^{-\mu_j t}+\frac{m_n\mu_n p_{nj}}{\mu_j}\right)\right]+c_i,\ i=\overline{1,n-1}$$
$$\frac{dv_n(t)}{dt}=\mu_n\sum_{j=1}^{n-1}p_{nj}b_{nj}\left[\sum_{i=1}^{n-1}\left(g_i e^{-\mu_i t}+\frac{m_n\mu_n p_{ni}}{\mu_i}\right)-K\right]+$$
$$+\sum_{j=1}^{n-1}\mu_j a_{jn}\left[g_j e^{-\mu_j t}+\frac{m_n\mu_n p_{nj}}{\mu_j}\right]+c_n$$

Integrate given ODE with initial conditions $v_i(0)=v_{i0},\ i=\overline{1,n}$, we will obtain

$$v_i(t)=\mu_n p_{ni}a_{ni}\sum_{j=1}^{n-1}\frac{g_j}{\mu_j}e^{-\mu_j t}+b_{in}g_i e^{-\mu_i t}+$$
$$+\left[\mu_n p_{ni}\left(Ka_{ni}-\mu_n m_n a_{ni}\sum_{j=1}^{n-1}\frac{p_{nj}}{\mu_j}-m_n b_{in}\right)+c_i\right]t+ \qquad (13)$$
$$-\mu_n p_{ni}a_{ni}\sum_{j=1}^{n-1}\frac{g_j}{\mu_j}-b_{in}g_i+v_{i0},\ i=\overline{1,n-1}$$

$$v_n(t) = -\mu_n \sum_{j=1}^{n-1} p_{nj} b_{nj} \sum_{i=1}^{n-1} \frac{g_i}{\mu_i} e^{-\mu_i t} - \sum_{j=1}^{n-1} a_{jn} g_j e^{-\mu_j t} +$$
$$+\left[\mu_n \left(m_n \mu_n \sum_{j=1}^{n-1} p_{nj} b_{nj} \sum_{i=1}^{n-1} \frac{p_{ni}}{\mu_i} - K \sum_{j=1}^{n-1} p_{nj} b_{nj} + m_n \sum_{j=1}^{n-1} p_{nj} a_{jn} \right) + c_n \right] t + \quad (14)$$
$$+\mu_n \sum_{i=1}^{n-1} p_{ni} b_{ni} \sum_{j=1}^{n-1} \frac{g_j}{\mu_j} + \sum_{j=1}^{n-1} a_{jn} g_j + v_{n0}$$

2. Numerical example

Let us examine the network which was described in the previous section with $n = 30$, $K = 61$, where K - the number of messages in the network. Service rates of messages in channels of network systems equal: $\mu_1 = \mu_{13} = \mu_{19} = \mu_{29} = 4$, $\mu_6 = 6$, $\mu_3 = \mu_9 = \mu_{10} = \mu_{16} = 5$, $\mu_4 = \mu_8 = \mu_{12} = \mu_{14} = \mu_{21} = \mu_{24} = \mu_{26} = \mu_{28} = 2$, $\mu_{25} = 1$, $\mu_{27} = 7$, $\mu_2 = \mu_5 = \mu_7 = \mu_{11} = \mu_{15} = \mu_{17} = \mu_{18} = \mu_{20} = \mu_{22} = \mu_{23} = 3$, $\mu_{30} = 40$, channel number in the central system – $m_{30} = 2$, probabilities of message transitions between network QS – $p_{30i} = 1/29$, $p_{i\,30} = 1$, $i = \overline{1,29}$, define also $p_{ii} = -1$, $i = \overline{1, 30}$, the rest of probabilities $p_{ij} = 0$, $i, j = \overline{1, 30}$. Let also $N_i(0) = 2$, $i = \overline{1,29}$, $N_{30}(0) = 3$. Charts of average number of messages in the QS are shown in Figures 2-5.

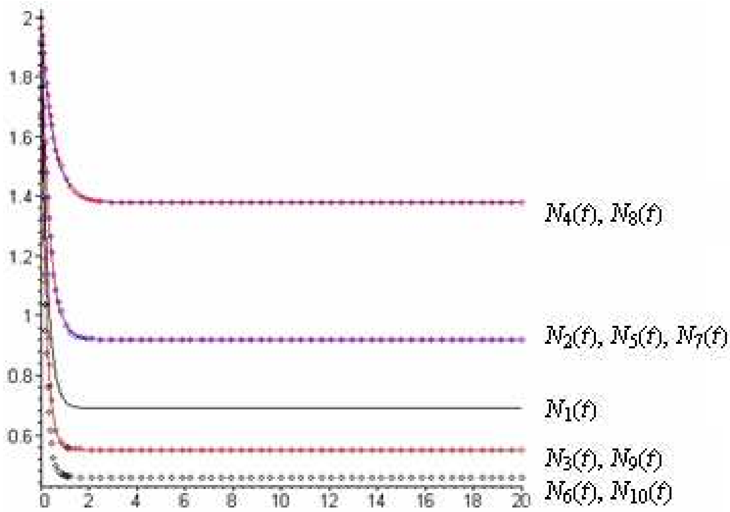

Fig. 2. Average number of messages in network systems S_i, $i = \overline{1,10}$

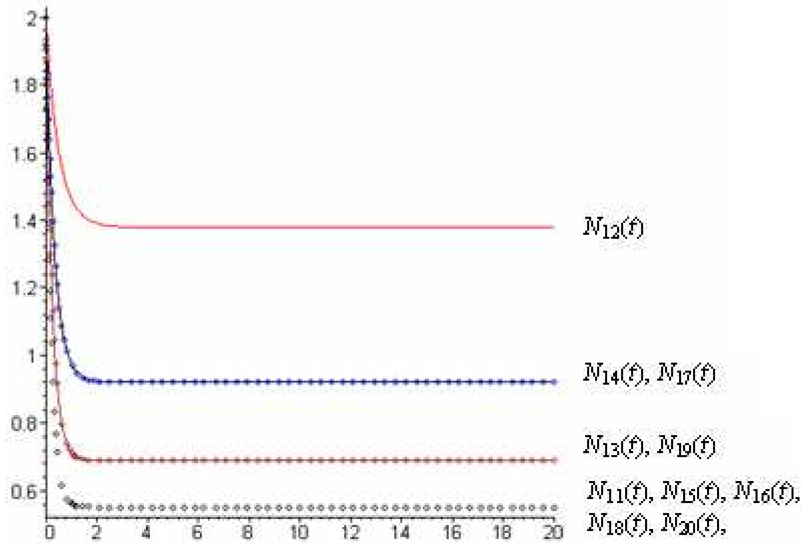

Fig. 3. Average number of messages in network systems S_i, $i=\overline{11, 20}$

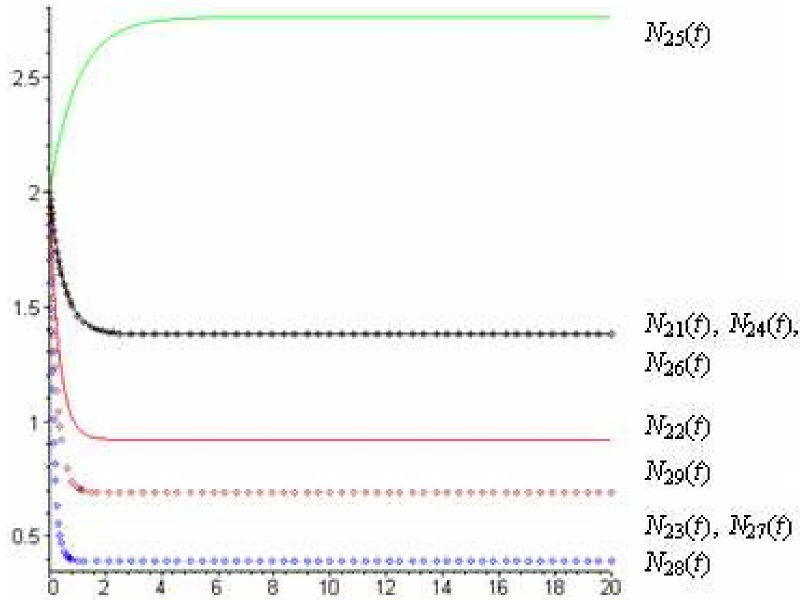

Fig. 4. Average number of messages in network systems S_i, $i=\overline{21, 29}$

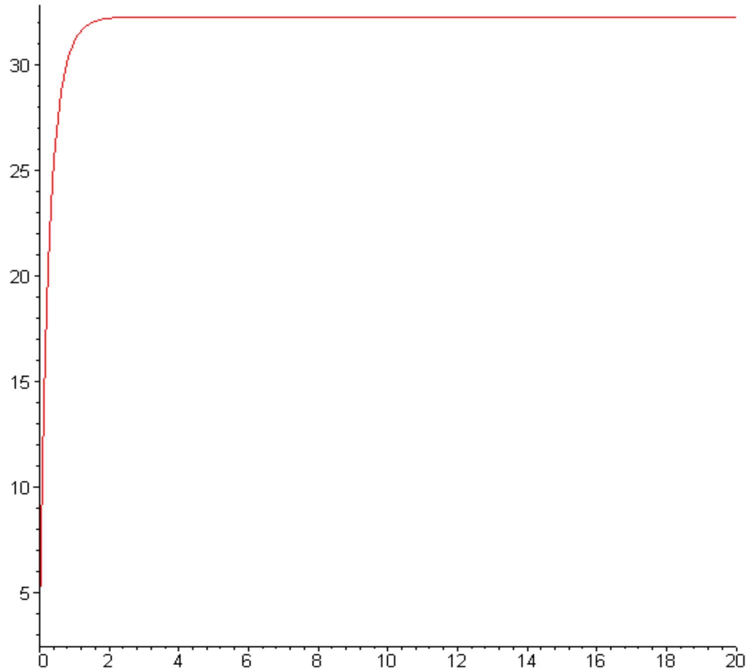

Fig. 5. Average number of messages in central QS

Specify values of m.e. of incomes from transitions between network states:

$$c_i = 20\cos\left(\frac{i\pi}{n} - \frac{\pi}{2}\right),\ i = \overline{1,30}$$

$$a_{30\,i} = 50\sin\frac{i\pi}{n},\ a_{i\,30} = 100\exp\left\{\frac{i}{n}\right\},\ i = \overline{1,29}$$

Then using relation (13), (14) with initial condition $v_i(0) = 100,\ i = \overline{1,29}$, $v_{30}(0) = 150$, the expressions for expected incomes of network systems were obtained. For example expression for expected income of central system is

$$v_{30}(t) = 795.4e^{-4t} + 2970.5e^{-3t} + 720.1e^{-5t} + 2354.9e^{-2t} + 149.7e^{-6t} - \\ - 823.7e^{-t} - 93.1e^{-7t} - 28654.2t + 15870.1$$

Charts of expected incomes of the network systems are shown in Figures 6-9.

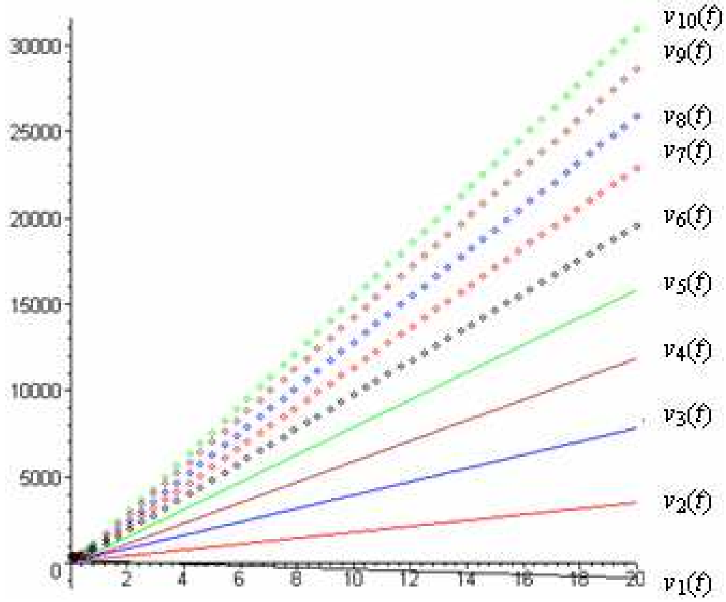

Fig. 6. Expected incomes of the systems S_i, $i = \overline{1, 10}$

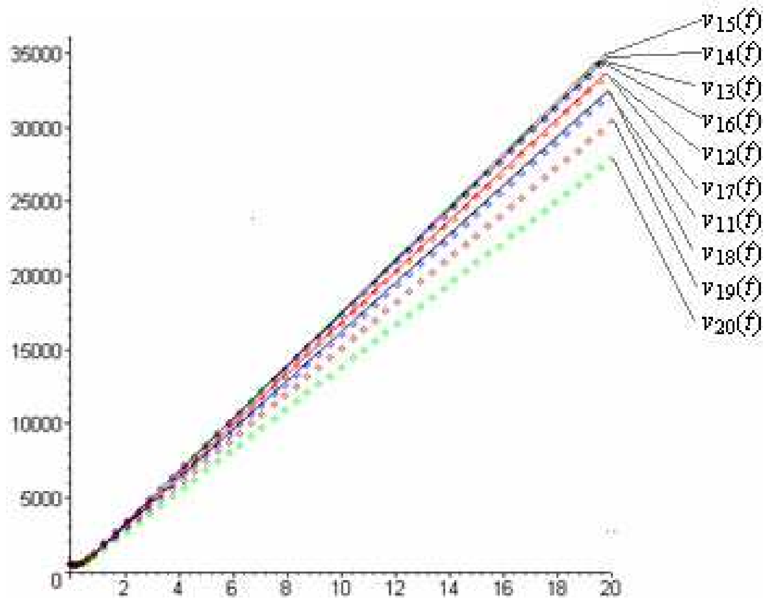

Fig. 7. Expected incomes of the systems S_i, $i = \overline{11, 20}$

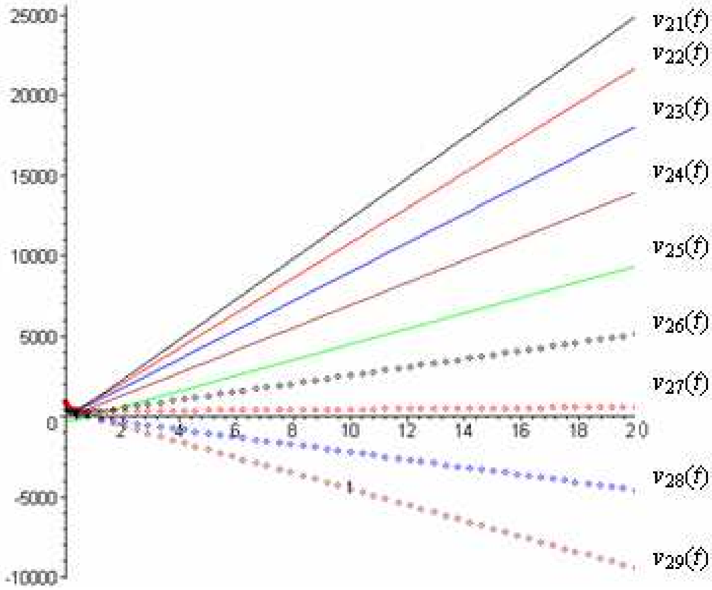

Fig. 8. Expected incomes of the systems S_i, $i = \overline{20, 29}$

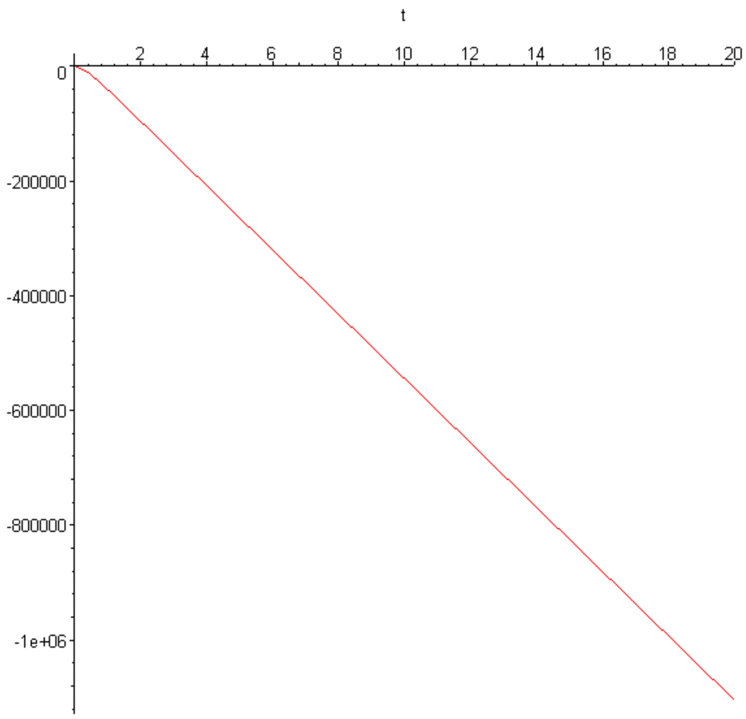

Fig. 9. Expected incomes of the central QS

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