NUMERICAL REALIZATION OF BOUNDARY ELEMENT METHOD FOR 1D FOURIER-KIRCHHOFF TYPE EQUATION

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Abstract. The 1D Fourier-Kirchhoff type equation in which the term connected with the first derivative of unknown function is considered. This equation supplemented by adequate boundary conditions is solved by means of the boundary element method. In the final part of the paper the examples of computations are shown.

1. Formulation of the problem

The one-dimensional Fourier-Kirchhoff type equation is considered

$$0 < x < L: \quad a \frac{\mathrm{d}^2 T(x)}{\mathrm{d} x^2} - u \frac{\mathrm{d} T(x)}{\mathrm{d} x} = 0 \tag{1}$$

where $a = \lambda/c$ is the thermal diffusivity (λ is the thermal conductivity and c is the volumetric specific heat, respectively), u is the velocity, T denotes the temperature and x is the spatial co-ordinate.

The equation (1) can be immidiatelly solved using the analytical methods, but taking into account the planned research concerning the numerical modeling of selected heat transfer problems the considerations presented below can be very useful at the next stages of investigations.

The equation (1) is supplemented by boundary conditions

$$\begin{aligned} x = 0: &\quad \Phi_1\left(T, \frac{\mathrm{d} T}{\mathrm{d} x} \right) = 0 \\ x = L: &\quad \Phi_2\left(T, \frac{\mathrm{d} T}{\mathrm{d} x} \right) = 0 \end{aligned} \tag{2}$$

To solve the problem (1), (2) the boundary element method is proposed.

2. Boundary element method

At first, the weighted residual criterion [1-3] for equation (1) is formulated

$$\int_0^L \left[\frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} - \frac{u}{a}\frac{\mathrm{d}T(x)}{\mathrm{d}x} \right] T^*(\xi, x)\,\mathrm{d}x = 0 \tag{3}$$

where ξ is the observation point and $T^*(\xi, x)$ is the fundamental solution.

The integral (3) we substitute by a sum of two integrals, while the first of them we transform by integrating twice by parts

$$\begin{gathered}\int_0^L \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} T^*(\xi, x)\,\mathrm{d}x = \left[T^*(\xi, x)\, \frac{\mathrm{d}T(x)}{\mathrm{d}x} \right]_0^L - \\ \int_0^L \frac{\mathrm{d}T(x)}{\mathrm{d}x}\, \frac{\partial T^*(\xi, x)}{\partial x}\,\mathrm{d}x = \\ \left[T^*(\xi, x)\frac{\mathrm{d}T(x)}{\mathrm{d}x} - \frac{\partial T^*(\xi, x)}{\partial x} T(x) \right]_0^L + \int_0^L \frac{\partial^2 T^*(\xi, x)}{\partial x^2} T(x)\,\mathrm{d}x = 0\end{gathered} \tag{4}$$

It is a special case of the well known 2nd Green formula which is used in derivations concerning the more complex (e.g. 2D) boundary integral equations. The second integral in (3) we integrate by parts

$$\begin{gathered}\int_0^L \frac{u}{a}\frac{\mathrm{d}T(x)}{\mathrm{d}x}\, T^*(\xi, x)\,\mathrm{d}x = \left[\frac{u}{a}\, T^*(\xi, x) T(x) \right]_{x=0}^{x=L} - \\ \int_0^L \frac{u}{a}\, \frac{\partial T^*(\xi, x)}{\partial x}\, T(x)\,\mathrm{d}x\end{gathered} \tag{5}$$

Introducing (4), (5) into (3) we have

$$\begin{gathered}\left[T^*(\xi, x)\frac{\mathrm{d}T(x)}{\mathrm{d}x} - \frac{\partial T^*(\xi, x)}{\partial x} T(x) - \frac{u}{a} T^*(\xi, x)\, T(x) \right]_0^L + \\ \int_0^L \left[\frac{\partial^2 T^*(\xi, x)}{\partial x^2} + \frac{u}{a}\, \frac{\partial T^*(\xi, x)}{\partial x} \right] T(x)\,\mathrm{d}x = 0\end{gathered} \tag{6}$$

Fundamental solution $T^*(\xi, x)$ should fulfill the following equation

$$0 < x < L: \quad \frac{\partial^2 T^*(\xi, x)}{\partial x^2} + \frac{u}{a}\, \frac{\partial T^*(\xi, x)}{\partial x} = -\delta(\xi, x) \tag{7}$$

where $\delta(\xi, x)$ is the Dirac function

$$\delta(\xi, x) = \begin{cases} 0, & \xi \neq x \\ \infty, & \xi = x \end{cases} \tag{8}$$

Taking into account the property (7) the equation (6) takes a form

$$\left[-\frac{1}{\lambda} T^*(\xi, x)\, q(x) + \frac{1}{\lambda} q^*(\xi, x) T(x) - \frac{u}{a} T^*(\xi, x) T(x) \right]_0^L + \\ -\int_0^L \delta(\xi, x) T(x)\, \mathrm{d}x = 0 \tag{9}$$

where

$$q(x) = -\lambda \frac{\mathrm{d}T(x)}{\mathrm{d}x} \\ q^*(\xi, x) = -\lambda \frac{\partial T^*(\xi, x)}{\partial x} \tag{10}$$

Finally, the equation (9) can be written as follows

$$T(\xi) + \left[\frac{1}{\lambda} T^*(\xi, x)\, q(x) \right]_0^L = \left[\left[\frac{1}{\lambda} q^*(\xi, x) - \frac{u}{a} T^*(\xi, x) \right] T(x) \right]_0^L \tag{11}$$

or

$$T(\xi) + \frac{1}{\lambda} T^*(\xi, L)\, q(L) - \frac{1}{\lambda} T^*(\xi, 0)\, q(0) = \\ \left[\frac{1}{\lambda} q^*(\xi, L) - \frac{u}{a} T^*(\xi, L) \right] T(L) - \left[\frac{1}{\lambda} q^*(\xi, 0) - \frac{u}{a} T^*(\xi, 0) \right] T(0) \tag{12}$$

The BEM resolving system is determined by calculations of the limits corresponding to the 'location' of observation point at the boundary (in this case we take $\xi \to 0^+$ and $\xi \to L^-$).

So, we obtain system of two equations which can be written in the matrix form

$$\begin{bmatrix} T(0) \\ T(L) \end{bmatrix} + \begin{bmatrix} -\frac{1}{\lambda} T^*(0, 0) & \frac{1}{\lambda} T^*(0, L) \\ -\frac{1}{\lambda} T^*(L, 0) & \frac{1}{\lambda} T^*(L, L) \end{bmatrix} \begin{bmatrix} q(0) \\ q(L) \end{bmatrix} = \\ \begin{bmatrix} -\left[\frac{1}{\lambda} q^*(0^+, 0) - \frac{u}{a} T^*(0, 0) \right] & \left[\frac{1}{\lambda} q^*(0^+, L) - \frac{u}{a} T^*(0, L) \right] \\ -\left[\frac{1}{\lambda} q^*(L^-, 0) - \frac{u}{a} T^*(L, 0) \right] & \left[\frac{1}{\lambda} q^*(L^-, L) - \frac{u}{a} T^*(L, L) \right] \end{bmatrix} \begin{bmatrix} T(0) \\ T(L) \end{bmatrix} \tag{13}$$

or

$$\begin{bmatrix} -\frac{1}{\lambda}\,T^*(0,0) & \frac{1}{\lambda}\,T^*(0,L) \\ -\frac{1}{\lambda}\,T^*(L,0) & \frac{1}{\lambda}\,T^*(L,L) \end{bmatrix} \begin{bmatrix} q(0) \\ q(L) \end{bmatrix} =$$
$$\begin{bmatrix} -\left[\frac{1}{\lambda}\,q^*(0^+,0) - \frac{u}{a}T^*(0,0)\right]-1 & \left[\frac{1}{\lambda}\,q^*(0^+,L) - \frac{u}{a}T^*(0,L)\right] \\ -\left[\frac{1}{\lambda}\,q^*(L^-,0) - \frac{u}{a}T^*(L,0)\right] & \left[\frac{1}{\lambda}\,q^*(L^-,L) - \frac{u}{a}T^*(L,L)\right]-1 \end{bmatrix} \begin{bmatrix} T(0) \\ T(L) \end{bmatrix} \tag{14}$$

In (14) four boundary values appear, this means $T(0)$, $T(L)$, $q(0)$, $q(L)$. Two of them are known from boundary conditions, the other two should be determined.

Next, the temperature T at the optional internal point $\xi \in (0, L)$ can be calculated using the formula (c.f. equation (12))

$$T(\xi) = \left[\frac{1}{\lambda}\,q^*(\xi,L) - \frac{u}{a}T^*(\xi,L)\right]T(L) - \left[\frac{1}{\lambda}\,q^*(\xi,0) - \frac{u}{a}T^*(\xi,0)\right]T(0) - \frac{1}{\lambda}\,T^*(\xi,L)\;q(L) + \frac{1}{\lambda}\,T^*(\xi,0)\;q(0) \tag{15}$$

3. Fundamental solution

To apply the boundary element method the form of fundamental solution should be known. For the case considered it is the following function [4]

$$T^*(\xi,x) = \frac{\operatorname{sgn}(x-\xi)}{2\lambda}\,\exp\left(-\frac{u}{a}\,(x-\xi)\right) - \frac{\Phi(x-\xi)}{\lambda} \tag{16}$$

where

$$\operatorname{sgn}(x-\xi) = \begin{cases} 1, & x-\xi > 0 \\ -1, & x-\xi < 0 \end{cases}, \qquad \Phi(x-\xi) = \begin{cases} 1, & x-\xi > 0 \\ 0, & x-\xi < 0 \end{cases} \tag{17}$$

To check does the solution (16) fulfills the equation (7) we calculate

$$\frac{\partial T^*(\xi,x)}{\partial x} = -\frac{u}{2\lambda a}\,\operatorname{sgn}(x-\xi)\exp\left(-\frac{u}{a}\,(x-\xi)\right) \tag{18}$$

and

$$\frac{\partial^2 T^*(\xi,x)}{\partial x^2} = \frac{u^2}{2\lambda a^2}\,\operatorname{sgn}(x-\xi)\exp\left(-\frac{u}{a}\,(x-\xi)\right) \tag{19}$$

Introducing (18), (19) into the left hand-side of equation (17) we obtain right hand-side of this equation. In Figure 1 the fundamental solution for $\xi = L/3$ is shown.

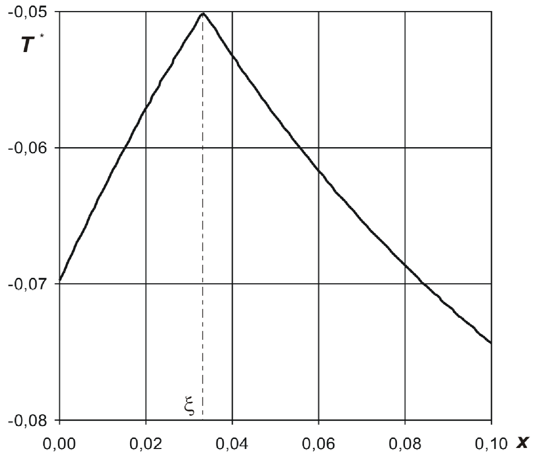

Fig. 1. Fundamental solution

The heat flux resulting from fundamental solution is equal to

$$q^*(\xi, x) = -\lambda \frac{\partial T^*(\xi, x)}{\partial x} = \frac{u}{2a} \operatorname{sgn}(x-\xi) \exp\left(-\frac{u}{a}(x-\xi)\right) \tag{20}$$

On the basis of (16) and (20) we have

$$\begin{aligned} &T^*(\xi, 0) = -\frac{1}{2\lambda} \exp\left(\frac{u}{a}\xi\right), && T^*(\xi, L) = \frac{1}{2\lambda} \exp\left(-\frac{u}{a}(L-\xi)\right) - \frac{1}{\lambda} \\ &T^*(0, 0) = -\frac{1}{2\lambda}, && T^*(0, L) = \frac{1}{2\lambda} \exp\left(-\frac{u}{a}L\right) - \frac{1}{\lambda} \\ &T^*(L, 0) = -\frac{1}{2\lambda} \exp\left(\frac{u}{a}L\right), && T^*(L, L) = -\frac{1}{2\lambda} \end{aligned} \tag{21}$$

and

$$\begin{aligned} &q^*(\xi, 0) = -\frac{u}{2a} \exp\left(\frac{u}{a}\xi\right), && q^*(\xi, L) = \frac{u}{2a} \exp\left(-\frac{u}{a}(L-\xi)\right) \\ &q^*(0, 0) = -\frac{u}{2a}, && q^*(0, L) = \frac{u}{2a} \exp\left(-\frac{u}{a}L\right) \\ &q^*(L, 0) = -\frac{u}{2a} \exp\left(\frac{u}{a}L\right), && q^*(L, L) = \frac{u}{2a} \end{aligned} \tag{22}$$

So, the system of equations (14) takes a form

$$\frac{1}{2\lambda^2}\begin{bmatrix} 1 & \exp\left(-\frac{uL}{a}\right)-2 \\ \exp\left(\frac{uL}{a}\right) & -1 \end{bmatrix}\begin{bmatrix} q(0) \\ q(L) \end{bmatrix} = \begin{bmatrix} -1 & \frac{u}{a\lambda} \\ 0 & \frac{u}{a\lambda}-1 \end{bmatrix}\begin{bmatrix} T(0) \\ T(L) \end{bmatrix} \tag{23}$$

while the equation (15) is as follows

$$T(\xi) = \frac{u}{a\lambda}T(L) + \left[\frac{1}{\lambda^2} - \frac{1}{2\lambda^2}\exp\left(-\frac{u}{a}\,(L-\xi)\right)\right]q(L) - \frac{1}{2\lambda^2}\exp\left(\frac{u}{a}\,\xi\right)q(0) \tag{24}$$

4. Results of computations

The layer of thickness L = 0.1 m is considered. In computations the following values of thermophysical parameters have been taken into account: thermal conductivity λ = 10 W/(mK), volumetric specific heat $c = 10^6$ W/(m^3 K), velocity u = 0.0001 m/s. It should be pointed out that the analytical solution for the problem considered is known

$$T(x) = C_1 + C_2 \exp\left(\frac{u}{a}\,x\right) \tag{25}$$

where C_1, C_2 are the integral constants. The values of C_1 and C_2 are calculated using boundary conditions, of course.

In the first version of computations the Dirichlet conditions T (0) = 50°C and T (L) = 100°C are assumed. In Figure 2 the temperature distribution obtained analytically (line) and by means of the BEM (symbols) is shown. Both results are the same. It confirms the exactness of the BEM.

The second version of computations is connected with Robin condition, this means for $x = L$: $q(L) = \alpha\,[T(L) - T_a]$, where α = 50 W/(m^2 K) is the heat transfer coefficient and T_a = 20°C is the ambient temperature. For x = 0 the Dirichlet condition T (0) = 50°C is assumed. In Figure 3 the comparison of analytical and numerical solutions is presented. As previously, the results are the same.

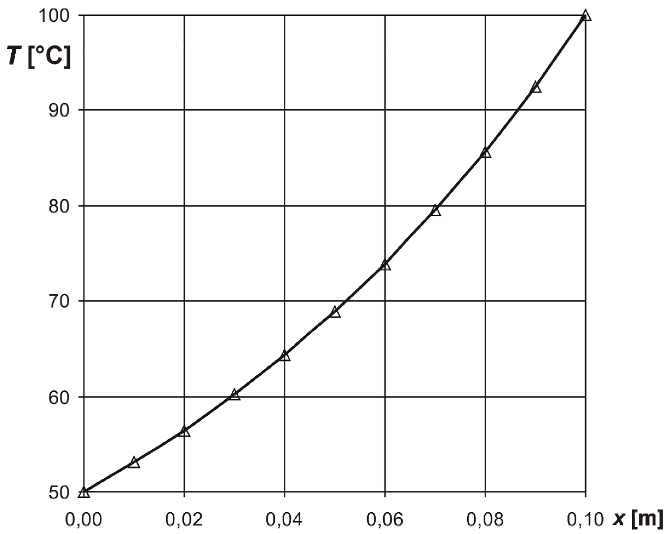

Fig. 2. Analytical and numerical solutions – variant 1

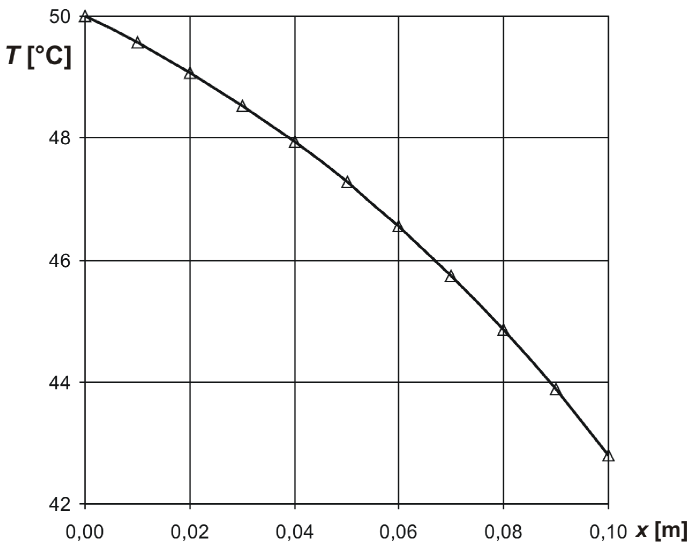

Fig. 3. Analytical and numerical solutions - variant 2

Final remarks

To solve the 1D Fourier-Kirchhoff type equation in which the term connected with the first derivative of unknown function appears the boundary element method is proposed. The results obtained compared with the analytical solutions confirm the exactness and effectiveness of the algorithm presented.

References

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