APPLICATION OF THE GREEN'S FUNCTION METHOD TO THE PROBLEM OF THERMALLY INDUCED VIBRATION OF A CIRCULAR PLATE

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Abstract. The thermally induced vibration of a homogeneous thin circular plate is considered. The plate is subjected to the activity of the point heat source which moves with constant angular velocity on the plate surface along a concentric circular trajectory. The thermal moment is derived on the basis of temperature field in the plate. The solution of the vibration problem is obtained by using the Green's function method.

Introduction

Changes in the temperature of a plate produce thermal stresses, which cause displacements of the plate. Cyclic changes in the temperature of plates induce transverse vibration. Several authors have studied the problem of thermally induced vibration of plates [1-5]. This problem has practical importance in mechanical, aeronautical and nuclear power industries. The thermally induced vibration of a circular and an annular plate is presented in the paper [1]. The plate is subjected to a sinusoidally varying heat flux on one surface and the other is thermally insulated. Applying the theory to circular and annular plates, the deflection, the stress distribution and the frequency response of the plates are calculated numerically.

In the paper [2] the thermally induced vibration of a simply supported and clamped circular plates is presented. In this analysis it is assumed that the distribution of temperature is linear through the thickness and along the radius. To solve this problem the author used an analytical method (the method of separation of variables) and the numerical method (FEM). The non-linear response of a thermally loaded isotropic plate is investigated in paper [3]. Authors excited the plate externally by a harmonic force near primary resonance and considered the in-plane thermal load to be axisymmetric. In paper [4] authors investigated the thermal deflection of an inverse thermoelastic problem in a thin isotropic circular plate. Authors determined temperature distribution and the thermal deflection on the curved surface of the plate by employing integral transform. The results are obtained in terms of series of Bessel's functions.

In this work, an analytical solution to the problem of thermally induced vibration of a circular plate is presented. The thermal moment caused by the temperature distribution on the thin circular plate is determined and displacements of the plate induced by the thermal moment are analyzed theoretically. The solution of the problem is obtained by using a time dependent Green's function.

1. Heat conduction problem

A circular isotropic plate of uniform thickness h and outer radius b (Fig. 1) is considered. This plate is heated by a heat source which moves on the plate surface along a concentric circular trajectory at radius r_0 with constant angular velocity ω.

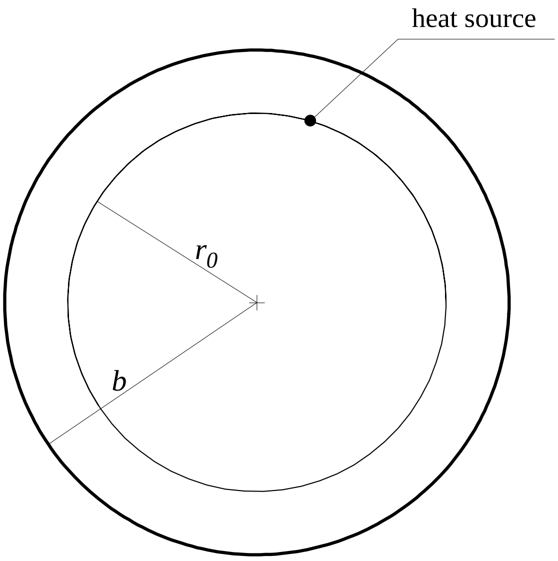

Fig. 1. A schema of a circular plate with a heat source

The temperature of the plate is governed by a heat conduction equation in cylindrical coordinates

$$\nabla^2 T + \frac{\partial^2 T}{\partial z^2} + \frac{1}{k} q(r,\phi,z,t) = \frac{1}{a}\frac{\partial T}{\partial t} \tag{1}$$

where: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$, $T(r,\phi,z,t)$ - temperature of the plate at the point (r,ϕ,z) at time t, k - thermal conductivity, a - thermal diffusivity and

$q(r,\phi,z,t)$ represents a heat generation term. The heat generation term is assumed in the form

$$q(r,\phi,z,t) = \theta\,\delta(r-r_0)\,\delta(\phi-\varphi(t))\,\delta(z-h) \tag{2}$$

where θ characterises the stream of the heat, $\delta(\;)$ is the Dirac delta function, $\varphi(t)$ is the function describing the movement of the heat source

$$\varphi(t) = \omega\, t \tag{3}$$

An analytical form of the temperature distribution in the considered plate have been given in paper [6] as a solution of the equation (1) with the following initial and boundary conditions:

$$T(r,\phi,z,0) = 0 \tag{4}$$

$$k\left.\frac{\partial T}{\partial r}\right|_{r=b} = -\alpha_0[T_0 - T(b,\phi,z,t)] \tag{5}$$

$$k\frac{\partial T}{\partial z}(r,\phi,h,t) = \alpha_0[T_0 - T(r,\phi,h,t)] \tag{6}$$

$$k\frac{\partial T}{\partial z}(r,\phi,0,t) = -\alpha_0[T_0 - T(r,\phi,0,t)] \tag{7}$$

where α_0 is the heat transfer coefficient, T_0 is the known temperature of the surrounding medium. The temperature for $T_0 = 0$, is expressed as (derivation is presented in the paper [6])

$$T(r,\phi,z,t) = \frac{8\,a\,\mu_0\,\Theta}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} A_{mnk}\,J(\gamma_{mk}r_0)\,J(\gamma_{mk}r)\,\Psi_n(z)\,\Psi_n(h)\,I_{mnk}(\phi,t) \tag{8}$$

where

$$A_{mnk} = \frac{\beta_n^2\,\gamma_{mk}^2}{\kappa_m\left(\beta_n^2+\mu_0^2\right)\left[2\mu_0 h\beta_n^2 + \left(\beta_n^2+\mu_0{}^2\right)\sin^2\beta_n h\right]\left[b^2\left(\gamma_{mk}^2+\mu_0^2\right)-m^2\right]J_m^2(\gamma_{mk}b)}$$

$$I_{mnk}(\phi,t) = \frac{1}{V_{mnk}^2 + m^2\omega^2}\Big[V_{mnk}\cos m(\phi-\omega t) - m\omega\sin m(\phi-\omega t)$$
$$- e^{-V_{mnk}t}\left(V_{mnk}\cos m\phi - m\omega\sin m\phi\right)\Big]\,,$$

$$V_{mnk} = a\left(\beta_n^2+\gamma_{mk}^2\right),\quad \mu_0 = \frac{\alpha_0}{k},\quad \kappa_m = 2 \text{ for } m=0 \text{ and } \kappa_m = 1 \text{ for } m \neq 0\,,$$

$$\psi_n(z) = \beta_n \cos\beta_n z + \mu_0 \sin\beta_n z, \quad n = 1,2,...$$

where β_n are roots of equation

$$2\mu_0\, \beta_n \cos\beta_n h - \left(\beta_n^2 - {\mu_0}^2\right)\sin\beta_n h = 0 \tag{9}$$

and γ_{mk} are roots of equation

$$b\gamma_{mk} J_{m-1}(b\gamma_{mk}) - (m + b\mu_0) J_m(b\gamma_{mk}) = 0 \tag{10}$$

2. The problem of thermally induced vibration of the plate

Thermally induced vibration of the considered plate is governed by the bi-harmonic differential equation [5]:

$$D\nabla^4 w + \mu \frac{\partial^2 w}{\partial t^2} = -\nabla^2 M_T \tag{11}$$

where D is a flexural stiffness, μ is a mass per unit area of the plate, $w(r,\Phi,t)$ is a displacement of the middle surface of the plate at point (r,Φ) at time t, and M_T denotes a thermal moment. The thermal moment appears as a result of temperature field in the plate and it is defined as [1]

$$M_T = \frac{\alpha E}{1-\nu} \int_0^h z T(r,\phi,z,t)\, dz \tag{12}$$

The presented study deals with the circular plate with simply supported edge which means that the following boundary conditions are satisfied

$$w = 0\,, \quad -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \phi^2}\right)\right] = 0 \quad \text{on } r = b \tag{13}$$

Moreover, the zero initial conditions are assumed

$$w = \frac{\partial w}{\partial t} = 0 \quad \text{for } t = 0 \tag{14}$$

Substituting (8) into equation (12) we obtain the thermal moment in the form

$$M_T(r,\phi,t) = \frac{8\, a\, \alpha\, E \mu_0\, \Theta}{(1-\nu)\pi} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} A_{mnk} B_n J(\gamma_{mk} r_0) J(\gamma_{mk} r) \Psi_n(h)\, I_{mnk}(\phi,t) \tag{15}$$

where

$$B_n = \frac{(1+h\mu_0)(\beta_n^2+\mu_0^2)\sin\beta_n h - 2\beta_n\mu_0}{2\beta_n^2\mu_0}$$

The solution of the problem in an analytical form is obtained by using the properties of the Green's function (GF). The GF for the vibration problem describes the displacement of the plate caused by impulse force. The GF function is a solution to the differential equation [7]:

$$D\nabla^4 G + \mu\frac{\partial^2 G}{\partial t^2} = -\frac{\delta(r-\rho)\delta(\phi-\psi)\delta(t-\tau)}{r} \tag{16}$$

Moreover, the Green's function satisfies the zero initial and homogeneous boundary conditions analogous to conditions (13). The solution of the vibration problem (11), (13) can be expressed as

$$w(r,\phi,t) = \int_0^t\int_0^b\int_0^{2\pi} \nabla^2 M_T(\rho,\psi,\tau)G(r,\phi,t;\rho,\psi,\tau)\,d\psi\,d\rho\,d\tau \tag{17}$$

3. The Green's function

The GF for the considered vibration problem may be written in the form of a series

$$G(r,\phi,t) = \sum_{m=-\infty}^{\infty} g_m(r,t)\cos m(\phi-\psi) \tag{18}$$

Substituting the series (18) into equation (16) and using the expansion [8]

$$\delta(\phi-\psi) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\cos m(\phi-\psi) \tag{19}$$

the differential equation for the functions $g_m(r,t)$ is obtained

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2}\right)^2 g_m + \frac{\mu}{D}\frac{\partial^2 g_m}{\partial t^2} = \frac{\delta(r-\rho)\,\delta(t-\tau)}{2\pi D\,r} \tag{20}$$

Next using (18) in boundary and initial conditions (13)-(14), we have

$$g_m(b,t) = 0\,, \quad \left[\frac{\partial^2 g_m}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial g_m}{\partial r} - \frac{m^2}{r^2}g_m\right)\right]\Bigg|_{r=b} = 0 \tag{21}$$

$$g_m(r,0)=0\,,\quad \left.\frac{\partial g_m}{\partial t}\right|_{t=0}=0 \tag{22}$$

The solution of the initial-boundary problem (11)-(13) can be presented in the form

$$g_m(r,t)=\sum_{n=1}^{\infty}R_{mn}(r)\Gamma_{mn}(t) \tag{23}$$

where $R_{mn}(r)$ are the eigenfunctions of the following boundary problem

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}\right]^2 R_{mn}(r)-\bar{\lambda}_{mn}^4 R_{mn}(r)=0 \tag{24}$$

$$R_{mn}(b)=0\,,\quad \left.\left[\frac{\partial^2 R_{mn}}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial R_{mn}}{\partial r}-\frac{m^2}{r^2}R_{mn}\right)\right]\right|_{r=b}=0 \tag{25}$$

The general solution of the differential equation (23) can be written in the form

$$R_{mn}(r)=\bar{C}_1 J_m(\bar{\lambda}_{mn}r)+\bar{C}_2 Y_m(\bar{\lambda}_{mn}r)+\bar{C}_3 I_m(\bar{\lambda}_{mn}r)+\bar{C}_4 K_m(\bar{\lambda}_{mn}r) \tag{26}$$

where J_m, Y_m are the Bessel functions of order *m*, and I_m, K_m are the modified Bessel functions of order *m*. As $|R_{mn}(0)|<+\infty$, so we assume

$$R_{mn}(r)=C_1 J_m\left(\lambda_{mn}\frac{r}{b}\right)+C_2 I_m\left(\lambda_{mn}\frac{r}{b}\right) \tag{27}$$

where $\lambda_{mn}=b\bar{\lambda}_{mn}$. Substituting the function (27) into the boundary conditions (25) we obtain a system of homogeneous equations

$$\begin{cases} C_1 J_m(\lambda_{mn})+C_2 I_m(\lambda_{mn})=0 \\ C_1 a_{21}+C_2 a_{22}=0 \end{cases} \tag{28}$$

where

$$a_{21}=\left(\lambda_{mn}^2-m(m-1)(1-\nu)\right)J_m(\lambda_{mn})-(1-\nu)\lambda_{mn}J_{m+1}(\lambda_{mn})$$

$$a_{22}=-\left(\lambda_{mn}^2+m(m-1)(1-\nu)\right)I_m(\lambda_{mn})+(1-\nu)\lambda_{mn}I_{m+1}(\lambda_{mn})$$

The non-trivial solution of the system exists for these λ_{mn} which satisfy the equation

$$2\lambda_{mn} J_m(\lambda_{mn}) I_m(\lambda_{mn}) - (1-\nu)\left[J_m(\lambda_{mn}) I_{m+1}(\lambda_{mn}) + J_{m+1}(\lambda_{mn}) I_m(\lambda_{mn})\right] = 0 \qquad (29)$$

Using (28a) we have

$$C_2 = -C_1 \frac{J_m(\lambda_{mn})}{I_m(\lambda_{mn})}$$

and the function R_{mn} can be written in the form

$$R_{mn}(r) = I_m(\lambda_{mn}) J_m\left(\lambda_{mn}\frac{r}{b}\right) - J_m(\lambda_{mn}) I_m\left(\lambda_{mn}\frac{r}{b}\right) \qquad (30)$$

Note that the functions R_{mn} satisfy the orthogonality condition

$$\int_0^b r\, R_{mn}(r) R_{mn'}(r)\, dr = \begin{cases} 0 & \text{for } n' \neq n \\ \chi_m(\lambda_{mn}) & \text{for } n' = n \end{cases} \qquad (31)$$

where

$$\chi_m(\lambda) = \frac{b^2}{2\lambda}\Big[J_m^2(\lambda) I_{m-1}(\lambda)\left(-\lambda I_{m-1}(\lambda) + 2(m-1) J_m(\lambda)\right)$$
$$I_m^2(\lambda) J_{m-1}(\lambda)\left(\lambda J_{m-1}(\lambda) - 2(m-1) J_m(\lambda)\right) + 2\lambda J_m^2(\lambda) I_m^2(\lambda)\Big] \qquad (32)$$

Taken into account (23), (24) and using the orthogonality condition (31) in equations (20) and (22), we obtain the differential equation

$$\frac{\partial^2 \Gamma_{mn}(t)}{\partial t^2} + \frac{D}{b^4 \mu}\lambda_{mn}^4 \Gamma_{mn}(t) = \frac{R_{mn}(\rho)}{2\pi\mu\,\chi_m(\lambda_{mn})}\delta(t-\tau) \qquad (33)$$

and initial conditions

$$\Gamma_{mn}(0) = 0\,, \qquad \left.\frac{d\,\Gamma_{mn}}{d\,t}\right|_{t=0} = 0 \qquad (34)$$

The solution of the initial problem (33), (34) has the form

$$\Gamma_{mn}(t) = \frac{R_{mn}(\rho)}{2\pi\mu\,\Omega_{mn}\,\chi_m(\lambda_{mn})}\sin\Omega_{mn}(t-\tau) H(t-\tau) \qquad (35)$$

where $\Omega^2_{mn} = \frac{D}{b^4 \mu} \lambda^4_{mn}$ and H denotes the Heaviside function.

Finally, on the basis of equations (18), (23) the Green's function for the simply supported circular plate can be written in the following form

$$G(r, \phi, t; \rho, \Psi, \tau) = \frac{H(t-\tau)}{2\pi\mu} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{R_{mn}(\rho)}{\Omega_{mn} \chi_m(\lambda_{mn})} R_{mn}(r) \sin \Omega_{mn}(t-\tau) \cos m(\phi - \psi) \tag{36}$$

Summary

In this paper the problem of the transverse vibration of a circular plate induced by a mobile heat source was considered. The formulation of the problem was based on the differential equations of heat conduction and transverse vibration of the plate, which were complemented by suitable initial and boundary conditions. The temperature distribution and transverse vibration of the circular plate in an analytical form were obtained by using the properties of the Green's function.

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