BOUNDARY ELEMENT METHOD FOR ELLIPTIC EQUATION - DETERMINATION OF FUNDAMENTAL SOLUTION

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Abstract. Elliptic equation with source term dependent on the first derivative of unknown function is considered. To solve this equation by means of the boundary element method the fundamental solution should be known. In the paper the fundamental solutions for 1D, 2D and 3D problems are derived.

Introduction

The following elliptic equation is considered

$$a\nabla^2 T(x) - \varepsilon\, u\, \frac{\partial T(x)}{\partial x_1} = 0 \tag{1}$$

where T is the temperature, $a=\lambda/c$ is the thermal diffusivity (λ is the thermal conductivity and c is the volumetric specific heat, respectively), u is the constant velocity and ε is the porosity. The 1D, 2D and 3D problems are analyzed, this means $x=\{x_1\}$, $x=\{x_1,x_2\}$, or $x=\{x_1,x_2,x_3\}$. In equation (1):

$$\nabla^2 T(x) = \sum_{e=1}^{M} \frac{\partial^2 T(x)}{\partial {x_e}^2} \tag{2}$$

where M is the problem dimension.

The equation (1) is supplemented by boundary conditions

$$\begin{array}{ll} x\in\Gamma_1: & T(x)=T_b \\ x\in\Gamma_2: & q(x) = -\lambda\,\dfrac{\partial T(x)}{\partial n} = q_b \end{array} \tag{3}$$

where T_b and q_b are the known boundary temperature and boundary heat flux, respectively, $\partial T/\partial n$ is the normal derivative

$$\frac{\partial T(x)}{\partial n} = \sum_{e=1}^{M} \frac{\partial T(x)}{\partial x_e} \cos \alpha_e \tag{4}$$

where $\cos \alpha_e$ are the directional cosines of the normal outward vector *n*.

The aim of investigations is to solve the problem formulated by means of the boundary element method. It is possible under the assumption that the fundamental solution is known. In this paper the fundamental solution is derived for 1D, 2D and 3D problems.

1. Boundary element method for elliptic equation with temperature - dependent source term

At first, the elliptic equation with temperature - dependent source term is considered

$$\lambda \nabla^2 T(x) - k\, T(x) = 0 \tag{5}$$

where k is the constant value.

The weighted residual criterion for equation (5) has the following form

$$\int_{\Omega} \left[\lambda \nabla^2 T(x) - k\, T(x) \right] T^*(\xi, x)\, \mathrm{d}\Omega = 0 \tag{6}$$

where $T^*(\xi, x)$ is the fundamental solution.

The fundamental solution should fulfill following equation

$$\lambda \nabla^2 T^*(\xi, x) - k\, T^*(\xi, x) = -\,\delta(\xi, x) \tag{7}$$

where $\delta(\xi, x)$ is the Dirac function.

It can be check that the following functions [1, 2]

$$T^*(\xi, x) = \begin{cases} \dfrac{1}{2\pi\lambda} \mathrm{K}_0\left(\sqrt{\dfrac{k}{\lambda}}\, r\right), & \text{2D problem} \\[2ex] \dfrac{1}{4\pi\lambda r} e^{-\sqrt{\frac{k}{\lambda}}\, r}, & \text{3D problem} \end{cases} \tag{8}$$

are the searched fundamental solutions.

In equation (8) r is the distance between the observation point ξ and the point x

$$r = \sqrt{\sum_{e=1}^{M} \left(x_e - \xi_e\right)^2} \tag{9}$$

Function $\mathrm{K}_0(\cdot)$ is the modified Bessel function of second kind, zero order [3].

It should be pointed out that in order to check the condition (7) for 2D problem the following dependences should be known [2, 3]

$$\frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{K}_0(z) = -\,\mathrm{K}_1(z) \tag{10}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{K}_1(z) = -\mathrm{K}_0(z) - \frac{1}{z}\,\mathrm{K}_1(z) \tag{11}$$

where $\mathrm{K}_1(\cdot)$ is the modified Bessel function of second kind, first order [2, 3].

The heat flux resulting from the fundamental solution is defined

$$q^*(\xi, x) = -\lambda\, \frac{\partial T^*(\xi, x)}{\partial n} \tag{12}$$

and this function can be calculated in analytical way

$$q^*(\xi, x) = \begin{cases} \dfrac{d\sqrt{k}}{2\pi\sqrt{\lambda}\, r}\ \mathrm{K}_1\left(\sqrt{\dfrac{k}{\lambda}}\ r\right), & \text{2D problem} \\[2ex] \dfrac{d}{4\pi r^2}\, e^{\left(-\sqrt{\frac{k}{\lambda}}\ r\right)} \left(\dfrac{1}{r} + \sqrt{\dfrac{k}{\lambda}}\right), & \text{3D problem} \end{cases} \tag{13}$$

where

$$d = \sum_{e=1}^{M} \left(x_e - \xi_e\right)\, \cos\alpha_e \tag{14}$$

After the mathematical transformations the equation (6) can be written in the form [2, 4]

$$\begin{aligned} &\int_{\Omega} \left[\lambda\, \nabla^2\, T^*(\xi, x) - k\, T^*(\xi, x)\right] T(x)\, \mathrm{d}\Omega \ + \\ &\int_{\Gamma} \lambda\ T^*(\xi, x)\ \frac{\partial T(x)}{\partial n}\ \mathrm{d}\Gamma \ - \ \int_{\Gamma} \lambda\ T(x)\ \frac{\partial T^*(\xi, x)}{\partial n}\ \mathrm{d}\Gamma = 0 \end{aligned} \tag{15}$$

Taking into account the formula (7) and introducing $q(x) = -\lambda\ \partial T(x)/\partial n$ and $q^*(\xi, x)$ one has the following boundary integral equation

$$B(\xi)\ T(\xi) + \int_\Gamma q(x)\ T^*(\xi, x)\,\mathrm{d}\,\Gamma = \int_\Gamma T(x)\ q^*(\xi, x)\,\mathrm{d}\,\Gamma \tag{16}$$

where for $\xi \in \Omega$: $B(\xi) = 1$, while for $\xi \in \Gamma$: $B(\xi) \in (0, 1)$ is the coefficient connected with the location of observation point ξ.

2. Fundamental solution for elliptic equation with source term dependent on temperature derivative

The weighted residual criterion for equation (1) has the following form

$$\int_\Omega \left[a\nabla^2 T(x) - \varepsilon\, u\, \frac{\partial T(x)}{\partial x_1} \right] T^*(\xi, x)\mathrm{d}\Omega = 0 \tag{17}$$

where $T^*(\xi, x)$ is the fundamental solution.

Using the second Green formula [2, 4] one has

$$\begin{gathered}\int_\Omega a\,\nabla^2\, T^*(\xi, x)\, T(x)\ \mathrm{d}\Omega + \int_\Gamma \left[a\, T^*(\xi, x) \frac{\partial T(x)}{\partial n} - a\, T(x) \frac{\partial T^*(\xi, x)}{\partial n} \right] \mathrm{d}\Gamma - \\ \int_\Omega \varepsilon\, u\, \frac{\partial T(x)}{\partial x_1} T^*(\xi, x)\mathrm{d}\Omega = 0\end{gathered} \tag{18}$$

or

$$\begin{gathered}\int_\Omega a\ \nabla^2\, T^*(\xi, x)\ T(x)\ \mathrm{d}\Omega - \frac{1}{c}\int_\Gamma T^*(\xi, x)\ q(x)\ \mathrm{d}\Gamma + \frac{1}{c}\int_\Gamma q^*(\xi, x)\ T(x)\ \mathrm{d}\Gamma + \\ \int_\Omega \varepsilon\, u\, \frac{\partial T^*(\xi, x)}{\partial x_1}\ T(x)\ \mathrm{d}\Omega - \int_\Gamma \varepsilon\, u\ T^*(\xi, x)\ T(x)\cos\alpha_1\,\mathrm{d}\Gamma = 0\end{gathered} \tag{19}$$

where $q(x) = -\lambda\ \partial T(x)/\partial n$ and $q^*(\xi, x) = -\lambda\ \partial T^*(\xi, x)/\partial n$, as previously.

Finally one obtains

$$\begin{gathered}\int_\Omega \left[a\ \nabla^2\, T^*(\xi, x) + \varepsilon\, u\, \frac{\partial T^*(\xi, x)}{\partial x_1} \right] T(x)\ \mathrm{d}\Omega - \frac{1}{c}\int_\Gamma T^*(\xi, x)\ q(x)\ \mathrm{d}\Gamma = \\ -\frac{1}{c}\int_\Gamma q^*(\xi, x)\ T(x)\ \mathrm{d}\Gamma + \int_\Gamma \varepsilon\, u\ T^*(\xi, x)\ T(x)\cos\alpha_1\ \mathrm{d}\Gamma\end{gathered} \tag{20}$$

It is visible that in the case considered the fundamental solution should fulfill following equation (c.f. equation (20))

$$a\,\nabla^2 T^*(\xi,x) + \varepsilon\,u\,\frac{\partial T^*(\xi,x)}{\partial x_1} = -\delta(\xi,x) \tag{21}$$

For 2D problem the solution of equation (21) is as follows [5]

$$T^*(\xi,x) = \frac{1}{2\pi\lambda}\,e^{-\frac{\varepsilon u (x_1-\xi_1)}{2a}}\,K_0\left(\frac{\varepsilon u r}{2a}\right) \tag{22}$$

It is true, because

$$\frac{\partial T^*(\xi,x)}{\partial x_1} = -\frac{\varepsilon u}{4\pi\lambda a}\,e^{-\frac{\varepsilon u (x_1-\xi_1)}{2a}}\left[K_0\left(\frac{\varepsilon u r}{2a}\right) + K_1\left(\frac{\varepsilon u r}{2a}\right)\frac{x_1-\xi_1}{r}\right] \tag{23}$$

$$\frac{\partial T^*(\xi,x)}{\partial x_2} = -\frac{\varepsilon u}{4\pi\lambda a}\,e^{-\frac{\varepsilon u (x_1-\xi_1)}{2a}}\,K_1\left(\frac{\varepsilon u r}{2a}\right)\frac{x_2-\xi_2}{r} \tag{24}$$

Next

$$\begin{aligned}\frac{\partial^2 T^*(\xi,x)}{\partial x_1^2} = -\frac{\varepsilon u}{4\pi\lambda a}\,e^{-\frac{\varepsilon u (x_1-\xi_1)}{2a}}\Bigg[&-\frac{\varepsilon u}{2a}\,K_0\left(\frac{\varepsilon u r}{2a}\right) - \\ &\frac{\varepsilon u}{a}\,K_1\left(\frac{\varepsilon u r}{2a}\right)\frac{x_1-\xi_1}{r} - \frac{\varepsilon u}{2a}\,K_0\left(\frac{\varepsilon u r}{2a}\right)\frac{(x_1-\xi_1)^2}{r^2} - \\ &2\,K_1\left(\frac{\varepsilon u r}{2a}\right)\frac{(x_1-\xi_1)^2}{r^3} + K_1\left(\frac{\varepsilon u r}{2a}\right)\frac{1}{r}\Bigg]\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial^2 T^*(\xi,x)}{\partial x_2^2} = -\frac{\varepsilon u}{4\pi\lambda a}\,e^{-\frac{\varepsilon u (x_1-\xi_1)}{2a}}\Bigg[&-\frac{\varepsilon u}{2a}\,K_0\left(\frac{\varepsilon u r}{2a}\right)\frac{(x_2-\xi_2)^2}{r^2} - \\ &2\,K_1\left(\frac{\varepsilon u r}{2a}\right)\frac{(x_2-\xi_2)^2}{r^3} + K_1\left(\frac{\varepsilon u r}{2a}\right)\frac{1}{r}\Bigg]\end{aligned} \tag{26}$$

Introducing formulas (23), (25), (26) into (21) for $x \neq \xi$ one obtains

$$
\begin{aligned}
& a\left\{-\frac{\varepsilon u}{4\pi\lambda a}\, e^{-\frac{\varepsilon u\,(x_1-\xi_1)}{2a}}\left[-\frac{\varepsilon u}{2a}\; K_0\!\left(\frac{\varepsilon u r}{2a}\right) -\right.\right. \\
& \frac{\varepsilon u}{a}\; K_1\!\left(\frac{\varepsilon u r}{2a}\right)\frac{x_1-\xi_1}{r} - \frac{\varepsilon u}{2a}\; K_0\!\left(\frac{\varepsilon u r}{2a}\right)\frac{(x_1-\xi_1)^2}{r^2} - \\
& \left. 2\,K_1\!\left(\frac{\varepsilon u r}{2a}\right)\frac{(x_1-\xi_1)^2}{r^3} + K_1\!\left(\frac{\varepsilon u r}{2a}\right)\frac{1}{r}\right] - \\
& \frac{\varepsilon u}{4\pi\lambda a}\, e^{-\frac{\varepsilon u\,(x_1-\xi_1)}{2a}}\left[-\frac{\varepsilon u}{2a}\; K_0\!\left(\frac{\varepsilon u r}{2a}\right)\frac{(x_2-\xi_2)^2}{r^2} - \right. \\
& \left.\left. 2\,K_1\!\left(\frac{\varepsilon u r}{2a}\right)\frac{(x_2-\xi_2)^2}{r^3} + K_1\!\left(\frac{\varepsilon u r}{2a}\right)\frac{1}{r}\right] + \right. \\
& \varepsilon u\left\{-\frac{\varepsilon u}{4\pi\lambda a}\, e^{-\frac{\varepsilon u\,(x_1-\xi_1)}{2a}}\left[K_0\!\left(\frac{\varepsilon u r}{2\,a}\right) + K_1\!\left(\frac{\varepsilon u r}{2\,a}\right)\frac{x_1-\xi_1}{r}\right]\right\} = 0
\end{aligned}
\tag{27}
$$

Additionally, the heat flux resulting from the fundamental solution (22) is determined (c.f. definition (12))

$$
\begin{aligned}
q^*(\xi,x) = \frac{\varepsilon u}{4\pi a}\, e^{-\frac{\varepsilon u\,(x_1-\xi_1)}{2a}} & \left\{\left[K_0\!\left(\frac{\varepsilon u r}{2\,a}\right) + K_1\!\left(\frac{\varepsilon u r}{2\,a}\right)\frac{x_1-\xi_1}{r}\right]\cos\alpha_1 + \right. \\
& \left. K_1\!\left(\frac{\varepsilon u r}{2\,a}\right)\frac{x_2-\xi_2}{r}\cos\alpha_2\right\}
\end{aligned}
\tag{28}
$$

For 1D and 3D problems fundamental solutions are unknown. Comparing the solutions for 2D problem (equations (8), (22)) it is visible, that the solution (22) is the product of solution (8) and the function $\exp\left(-\frac{\varepsilon u\,(x_1-\xi_1)}{2a}\right)$. So, for 3D problem the following fundamental solution is proposed

$$
T^*(\xi,x) = \frac{1}{4\pi\lambda r}\, e^{-\frac{\varepsilon u}{2a}\left[r+(x_1-\xi_1)\right]}
\tag{29}
$$

We check: does the solution (29) fulfills the equation (21)?
So

$$\frac{\partial T^*(\xi,x)}{\partial x_1} = \frac{1}{4\pi\lambda r} e^{-\frac{\varepsilon u}{2a}[r+(x_1-\xi_1)]}\left[-\frac{x_1-\xi_1}{r^2} - \frac{\varepsilon u}{2a}\frac{x_1-\xi_1}{r} - \frac{\varepsilon u}{2a}\right] \quad (30)$$

$$\frac{\partial T^*(\xi,x)}{\partial x_2} = \frac{1}{4\pi\lambda r} e^{-\frac{\varepsilon u}{2a}[r+(x_1-\xi_1)]}\left[-\frac{x_2-\xi_2}{r^2} - \frac{\varepsilon u}{2a}\frac{x_2-\xi_2}{r}\right] \quad (31)$$

$$\frac{\partial T^*(\xi,x)}{\partial x_3} = \frac{1}{4\pi\lambda r} e^{-\frac{\varepsilon u}{2a}[r+(x_1-\xi_1)]}\left[-\frac{x_3-\xi_3}{r^2} - \frac{\varepsilon u}{2a}\frac{x_3-\xi_3}{r}\right] \quad (32)$$

and

$$\begin{aligned}\frac{\partial^2 T^*(\xi,x)}{\partial x_1^2} = \frac{1}{4\pi\lambda r} e^{-\frac{\varepsilon u}{2a}[r+(x_1-\xi_1)]}\Bigg[\, & 3\frac{(x_1-\xi_1)^2}{r^4} + \\ & \left(\frac{\varepsilon u}{2a}\right)^2\frac{(x_1-\xi_1)^2}{r^2} + \left(\frac{\varepsilon u}{2a}\right)^2 + \frac{3\varepsilon u}{2a}\frac{(x_1-\xi_1)^2}{r^3} + \\ & \frac{\varepsilon u}{a}\frac{x_1-\xi_1}{r^2} + 2\left(\frac{\varepsilon u}{2a}\right)^2\frac{x_1-\xi_1}{r} - \frac{1}{r^2} - \frac{\varepsilon u}{2a}\frac{1}{r}\Bigg]\end{aligned} \quad (33)$$

$$\begin{aligned}\frac{\partial^2 T^*(\xi,x)}{\partial x_2^2} = \frac{1}{4\pi\lambda r} e^{-\frac{\varepsilon u}{2a}[r+(x_1-\xi_1)]}\Bigg[\, & 3\frac{(x_2-\xi_2)^2}{r^4} + \\ & \frac{3\varepsilon u}{2a}\frac{(x_2-\xi_2)^2}{r^3} + \left(\frac{\varepsilon u}{2a}\right)^2\frac{(x_2-\xi_2)^2}{r^2} - \frac{1}{r^2} - \frac{\varepsilon u}{2a}\frac{1}{r}\Bigg]\end{aligned} \quad (34)$$

$$\begin{aligned}\frac{\partial^2 T^*(\xi,x)}{\partial x_3^2} = \frac{1}{4\pi\lambda r} e^{-\frac{\varepsilon u}{2a}[r+(x_1-\xi_1)]}\Bigg[\, & 3\frac{(x_3-\xi_3)^2}{r^4} + \\ & \frac{3\varepsilon u}{2a}\frac{(x_3-\xi_3)^2}{r^3} + \left(\frac{\varepsilon u}{2a}\right)^2\frac{(x_3-\xi_3)^2}{r^2} - \frac{1}{r^2} - \frac{\varepsilon u}{2a}\frac{1}{r}\Bigg]\end{aligned} \quad (35)$$

Introducing (30), (33), (34), (35) into (21) for $x \neq \xi$ one obtains

$$a \left\{ \frac{1}{4\pi\lambda r} e^{-\frac{\varepsilon u}{2a}[r+(x_1-\xi_1)]} \left[3\frac{(x_1-\xi_1)^2}{r^4} + \left(\frac{\varepsilon u}{2a}\right)^2 \frac{(x_1-\xi_1)^2}{r^2} + \left(\frac{\varepsilon u}{2a}\right)^2 + \right.\right.$$

$$+ \frac{3\varepsilon u}{2a}\frac{(x_1-\xi_1)^2}{r^3} + \frac{\varepsilon u}{a}\frac{x_1-\xi_1}{r^2} + 2\left(\frac{\varepsilon u}{2a}\right)^2 \frac{x_1-\xi_1}{r} - \frac{1}{r^2} - \frac{\varepsilon u}{2a}\frac{1}{r} \Big] +$$

$$\frac{1}{4\pi\lambda r} e^{-\frac{\varepsilon u}{2a}[r+(x_1-\xi_1)]} \left[3\frac{(x_2-\xi_2)^2}{r^4} + \frac{3\varepsilon u}{2a}\frac{(x_2-\xi_2)^2}{r^3} + \right.$$

$$\left(\frac{\varepsilon u}{2a}\right)^2 \frac{(x_2-\xi_2)^2}{r^2} - \frac{1}{r^2} - \frac{\varepsilon u}{2a}\frac{1}{r} \Big] + \frac{1}{4\pi\lambda r} e^{-\frac{\varepsilon u}{2a}[r+(x_1-\xi_1)]} \left[3\frac{(x_3-\xi_3)^2}{r^4} + \right. \tag{36}$$

$$\frac{3\varepsilon u}{2a}\frac{(x_3-\xi_3)^2}{r^3} + \left(\frac{\varepsilon u}{2a}\right)^2 \frac{(x_3-\xi_3)^2}{r^2} - \frac{1}{r^2} - \frac{\varepsilon u}{2a}\frac{1}{r} \Big] +$$

$$\varepsilon u \left\{ \frac{1}{4\pi\lambda r} e^{-\frac{\varepsilon u}{2a}[r+(x_1-\xi_1)]} \left[-\frac{x_1-\xi_1}{r^2} - \frac{\varepsilon u}{2a}\frac{x_1-\xi_1}{r} - \frac{\varepsilon u}{2a} \right] \right\} = 0$$

For 3D problem the function $q^*(\xi, x)$ has the following form

$$q^*(\xi, x) = \frac{1}{4\pi r} e^{-\frac{\varepsilon u}{2a}[r+(x_1-\xi_1)]} \left[\left(\frac{x_1-\xi_1}{r^2} + \frac{\varepsilon u}{2a}\frac{x_1-\xi_1}{r} + \frac{\varepsilon u}{2a} \right) \cos\alpha_1 + \right.$$
$$\left. \left(\frac{x_2-\xi_2}{r^2} + \frac{\varepsilon u}{2a}\frac{x_2-\xi_2}{r} \right) \cos\alpha_2 + \left(\frac{x_3-\xi_3}{r^2} + \frac{\varepsilon u}{2a}\frac{x_3-\xi_3}{r} \right) \cos\alpha_3 \right] \tag{37}$$

For 1D problem the equation (21) takes a form

$$a\frac{\partial^2 T^*(\xi, x)}{\partial x^2} + \varepsilon u \frac{\partial T^*(\xi, x)}{\partial x} = -\delta(\xi, x) \tag{38}$$

where $x = x_1$.

To determine the fundamental solution at first the following equation is considered

$$a\,\frac{d^2T^*(x)}{dx^2} + \varepsilon u\,\frac{dT^*(x)}{dx} = 0 \tag{39}$$

The substitution is introduced

$$T^*(x) = e^{\tau x} \tag{40}$$

where τ is a constant value.

On the basis of (40) the following fundamental solution of equation (1) for 1D problem is proposed

$$T^*(\xi,x) = \begin{cases} -\dfrac{1}{2\lambda}\,e^{-\frac{\varepsilon u}{a}(x-\xi)}, & x-\xi<0 \\[2ex] \dfrac{1}{2\lambda}\,e^{-\frac{\varepsilon u}{a}(x-\xi)} - \dfrac{1}{\lambda}, & x-\xi>0 \end{cases} \tag{41}$$

Additionally, the heat flux resulting from fundamental solution

$$q^*(\xi,x) = -\lambda\,\frac{dT^*(\xi,x)}{dx} \tag{42}$$

is calculated

$$q^*(\xi,x) = \frac{\varepsilon u}{2a}\,\mathrm{sgn}(x-\xi)\,e^{-\frac{\varepsilon u}{a}(x-\xi)} \tag{43}$$

Summing up, fundamental solutions for equation (1) are the following

$$T^*(\xi,x) = \begin{cases} -\dfrac{1}{2\lambda}\,e^{-\frac{\varepsilon u}{a}(x-\xi)}, & x-\xi<0 \\[2ex] \dfrac{1}{2\lambda}\,e^{-\frac{\varepsilon u}{a}(x-\xi)} - \dfrac{1}{\lambda}, & x-\xi>0 \end{cases}, \quad \text{1D problem}$$

$$T^*(\xi,x) = \begin{cases} \dfrac{1}{2\pi\lambda}\,e^{-\frac{\varepsilon u\,(x_1-\xi_1)}{2a}}\,K_0\!\left(\dfrac{\varepsilon u r}{2a}\right), & \text{2D problem} \\[2ex] \dfrac{1}{4\pi\lambda r}\,e^{-\frac{\varepsilon u}{2a}\left[r+(x_1-\xi_1)\right]}, & \text{3D problem} \end{cases} \tag{44}$$

while the heat fluxes resulting from the fundamental solutions are of the form

$$q^*(\xi,x) = \frac{\varepsilon u}{2a}\,\mathrm{sgn}(x-\xi)\,e^{-\frac{\varepsilon u}{a}(x-\xi)}, \qquad \text{1D problem}$$

$$q^*(\xi,x) = \frac{\varepsilon u}{4\pi a}\,e^{-\frac{\varepsilon u (x_1-\xi_1)}{2a}}\left\{\left[K_0\left(\frac{\varepsilon u r}{2a}\right) + K_1\left(\frac{\varepsilon u r}{2a}\right)\frac{x_1-\xi_1}{r}\right]\cos\alpha_1 + K_1\left(\frac{\varepsilon u r}{2a}\right)\frac{x_2-\xi_2}{r}\cos\alpha_2\right\}, \qquad \text{2D problem} \tag{45}$$

$$q^*(\xi,x) = \frac{1}{4\pi r}\,e^{-\frac{\varepsilon u}{2a}[r+(x_1-\xi_1)]}\left[\left(\frac{x_1-\xi_1}{r^2} + \frac{\varepsilon u}{2a}\frac{x_1-\xi_1}{r} + \frac{\varepsilon u}{2a}\right)\cos\alpha_1 + \left(\frac{x_2-\xi_2}{r^2} + \frac{\varepsilon u}{2a}\frac{x_2-\xi_2}{r}\right)\cos\alpha_2 + \left(\frac{x_3-\xi_3}{r^2} + \frac{\varepsilon u}{2a}\frac{x_3-\xi_3}{r}\right)\cos\alpha_3\right], \quad \text{3D problem}$$

Conclusions

It is well known, that the boundary element method can be applied for numerical solution of optional equation under the assumption that the fundamental solution is known. In the paper the fundamental solutions for 1D, 2D and 3D problems for elliptic equation with source term dependent on the first derivative of unknown function are derived.

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