FREE TRANSVERSE VIBRATIONS OF NON-UNIFORM BEAMS

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Abstract. In this paper the Green's function method for the free vibration problem of non-uniform Bernoulli-Euler beams is presented. To find the Green's function of the fourth order differential operator, occurring at the beam's equation of motion, the power series method is proposed.

Introduction

The approximate approach for the vibration problem of non-uniform Bernoulli Euler beams was presented in [1, 2]. The authors of these publications used the Rayleigh-Ritz method for beams with a polynomial cross section and moment of inertia with additional discrete elements [1] and for different boundary conditions [6]. An analytical solution of the free vibration problems of the non-uniform beams can only be found for some special types of the beams. The closed form solutions of the free vibration problem for second and fourth order polynomial parameters characterizing beams have been found in [3, 4]. The Green's function method was presented in papers [4-6].

1. Formulation and solution of the problem

Let's consider a non-uniform beam length L with cross section area $A(x)$ and moment of inertia $I(x)$ carrying any number of discrete elements.

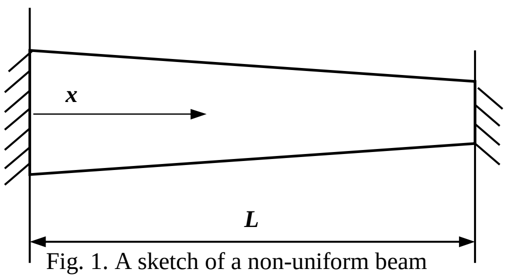

Fig. 1. A sketch of a non-uniform beam

According to the Bernoulli-Euler theory the following equation governs a free vibration of the beam with attached discrete elements (springs, masses, oscillators) [5]:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 Y(x,t)}{\partial x^2}\right]+\rho A(x)\frac{\partial^2 Y(x,t)}{\partial t^2}=\mathbf{F}\left[Y(x,t)\right] \tag{1}$$

where Y is the function of deflection, ρ is the mass density of the beam material, E is the modulus of elasticity and the form of operator $\mathbf{F}$ depends on the nature of the attached discrete systems. Function Y satisfies homogeneous boundary conditions

$$\mathbf{B}_0\left[Y(x,t)\right]\Big|_{x=0}=0, \quad \mathbf{B}_1\left[Y(x,t)\right]\Big|_{x=L}=0 \tag{2}$$

Natural frequencies of the beam are harmonic $Y(x,t)=\bar{Y}(x)e^{i\omega t}$, therefore equations (1) and (2) may be written in the form:

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\bar{Y}(x)}{dx^2}\right]-\rho\omega^2 A(x)\bar{Y}(x)=\bar{\mathbf{F}}\left[\bar{Y}(x)\right] \tag{3}$$

$$\bar{\mathbf{B}}_0\left[\bar{Y}(x)\right]\Big|_{x=0}=0, \quad \bar{\mathbf{B}}_1\left[\bar{Y}(x)\right]\Big|_{x=L}=0 \tag{4}$$

By introducing non-dimensional coordinates and values: $\xi=\frac{x}{L}$, $y=\frac{\bar{Y}}{L}$, $\Omega^4=\frac{\rho A_0\omega^2 L^4}{EI_0}$, where ω is the natural vibration of the beam, we obtain equations (3)-(4) in the form:

$$\frac{d^2}{d\xi^2}\left[I(\xi)\frac{d^2 y(\xi)}{d\xi^2}\right]-\Omega^4 A(\xi)y(\xi)=\tilde{\mathbf{F}}\left[y(\xi)\right] \tag{5}$$

$$\tilde{\mathbf{B}}_0\left[y(\xi)\right]\Big|_{\xi=0}=0, \quad \tilde{\mathbf{B}}_1\left[y(\xi)\right]\Big|_{\xi=1}=0 \tag{6}$$

The solution to this problem (5)-(6) can be obtained with the use of the Green's function method and properties of the Green's function. If the Green's function of the linear differential operator $\mathbf{L}=\frac{d^2}{d\xi^2}\left[I(\xi)\frac{d^2}{d\xi^2}\right]-\Omega^4 A(\xi)$ is known, the solution to the problem (5)-(6) may be written as follows:

$$y(\xi)=\int_0^1 G(\xi,\zeta)\mathbf{F}\left[y(\zeta)\right]d\zeta \tag{7}$$

Equation (7) is used in the analysis of the vibration of the beam. It yields to the frequency equation which is solved numerically with respect to the eigenfrequencies. The eigenfunctions corresponding to the eigenfrequencies are derived by using equation (7).

2. Green's function of the fourth order differential operator

Let's consider the following operator [4]:

$$\mathbf{L} \equiv \frac{d^2}{d\xi^2}\left(p_3(\xi)\frac{d^2}{d\xi^2}\right) + \frac{d}{d\xi}\left(p_2(\xi)\frac{d}{d\xi}\right) - \Omega^4 p_1(\xi) \tag{8}$$

The Green's function of this operator (8) satisfies the equation:

$$\mathbf{L}\left[G(\xi,\zeta)\right] = \delta(\xi-\zeta) \tag{9}$$

where δ is the Dirac delta function. Function G has the form:

$$G(\xi,\zeta) = G_0(\xi,\zeta) + G_1(\xi,\zeta) H(\xi-\zeta) \tag{10}$$

H is a unit step function, G_0 and G_1 are solutions to a homogeneous equation

$$\mathbf{L}\left[G(\xi,\zeta)\right] = 0 \tag{11}$$

G_1 also satisfies the following conditions:

$$G_1(\xi,\zeta)\Big|_{\xi=\zeta} = \frac{\partial G_1(\xi,\zeta)}{\partial \xi}\Bigg|_{\xi=\zeta} = \frac{\partial^2 G_1(\xi,\zeta)}{\partial \xi^2}\Bigg|_{\xi=\zeta} = 0, \quad \frac{\partial^3 G_1(\xi,\zeta)}{\partial \xi^3}\Bigg|_{\xi=\zeta} = \frac{1}{p_3(\zeta)} \tag{12}$$

Assuming that $p_i(\xi) = \sum_{r=0}^{\infty} \frac{p_{i,r}}{r!}\xi^r$ for i = 1, 2, 3, we are looking for the solution to $\mathbf{L}\left[V(\xi)\right] = 0$ with the use of the power series method also as the sum $V(\xi) = \sum_{r=0}^{\infty}\frac{v_r}{r!}\xi^r$. Substituting p_i and V into a homogeneous equation leads to:

$$\sum_{r=0}^{\infty} \frac{P_{3,r+2} + P_{2,r+1} - \Omega^4 P_{1,r}}{r!}\cdot \xi^r = 0 \tag{13}$$

where $P_{i,r}=\sum_{j=0}^{r}\binom{r}{j}p_{i,j}v_{r+i-j}$, i = 1, 2, 3. We may write this equation (13) as follows:

$$\sum_{j=0}^{r+2}\binom{r+2}{j}p_{3,j}v_{r+4-j}+\sum_{j=0}^{r+1}\binom{r+1}{j}p_{2,j}v_{r+2-j}-\Omega^4\sum_{j=0}^{r}\binom{r}{j}p_{1,j}v_{r-j}=0 \tag{14}$$

From (14) we can determinate values of unknown coefficients v_{r+4} (r = 0, 1, 2,...) by means of factors v_0, v_1, v_2, v_3:

$$v_{r+4}=-\frac{1}{p_{3,0}}\left[\sum_{j=1}^{r+2}\binom{r+2}{j}p_{3,j}v_{r+4-j}+\sum_{j=0}^{r+1}\binom{r+1}{j}p_{2,j}v_{r+2-j}-\Omega^4\sum_{j=0}^{r}\binom{r}{j}p_{1,j}v_{r-j}\right] \tag{15}$$

Equation (11) has four linear independent solutions:

$$V_k^*(\xi)=\sum_{r=0}^{\infty}\frac{v_{k,r}^*}{r!}\xi^r \qquad k = 1, 2, 3, 4 \tag{16}$$

To find them we must assume that functions V_k^* satisfy conditons:

$$\left.\frac{d^iV^*(\xi)}{d\xi^i}\right|_{\xi=0}=\delta_{k,i+1} \qquad i = 0, 1, 2, 3; \quad k = 1, 2, 3, 4 \tag{17}$$

where $\delta_{k,i+1}$ is the Kronecker delta function. Because of $\left.\frac{d^iV^*(\xi)}{d\xi^i}\right|_{\xi=0}=v_{k,i}^*$, we can write (17) in the form

$$v_{k,i}^*=\delta_{k,i+1} \quad i = 0, 1, 2, 3; \quad k = 1, 2, 3, 4 \tag{18}$$

With the use of formula (18) and equation (14) other coefficients of function V_k^* are calculated:

$$v_{k,r+4}^*=-\frac{1}{p_{3,0}}\left[\sum_{j=1}^{r+2}\binom{r+2}{j}p_{3,j}v_{k,r+4-j}^*+\sum_{j=0}^{r+1}\binom{r+1}{j}p_{2,j}v_{k,r+2-j}^*-\Omega^4\sum_{j=0}^{r}\binom{r}{j}p_{1,j}v_{k,r-j}^*\right]$$

$$k = 1, 2, 3, 4; \quad r = 0, 1, 2,... \tag{19}$$

Finally, a general solution to the homogeneous equation $\mathbf{L}\left[V(\xi)\right]=0$ has the form:

$$V(\xi)=\sum_{k=1}^{4}C_kV_k^*(\xi)=\sum_{k=1}^{4}C_k\left[\sum_{r=0}^{\infty}\frac{v_{k,r}^*}{r!}\xi^r\right] \tag{20}$$

where $v_{k,r}^*$ are expressed by (18) and (19).

To calculate a particular solution $G_1(\xi,\zeta)H(\xi-\zeta)$ to equation (9) it´s necessary to have in mind condition (12) where $p_3(\zeta)=\sum_{r=0}^{\infty}\frac{p_{3,r}}{r!}\zeta^r$. Using them we obtain the function $G_1(\xi,\zeta)$ as follows:

$$G_1(\xi,\zeta)=\sum_{k=1}^{4}\bar{C}_k(\zeta)V_k^*(\xi) \tag{21}$$

where $\bar{C}_k(\zeta)=\frac{(-1)^k W_k}{p_3(\zeta)W}$, $W=\det\left[w_{ij}(\zeta)\right]_{i,j=1,2,3,4}$, $W_k=\det\left[w_{ij}(\zeta)\right]_{\substack{i=1,2,3\\ j\neq k}}$ for j, k = 1, 2, 3, 4 and $w_{ij}(\zeta)=\frac{d^{i-1}V_j^*(\zeta)}{d\zeta^{i-1}}$

In result, the Green's function $G(\xi,\zeta)$ of the differential operator **L** can be presented in the form:

$$G(\xi,\zeta)=\sum_{k=1}^{4}C_kV_k^*(\xi)+H(\xi-\zeta)\cdot\sum_{k=1}^{4}\frac{(-1)^k W_k}{p_3(\zeta)W}V_k^*(\xi) \tag{22}$$

Unknown constants C_k are calculated with the use of boundary conditions (6). For example, the system of equations

$$\sum_{k=1}^{4}C_kV_k^{*(i+2)}(0)=0\,,\quad \sum_{k=1}^{4}\left[C_k+\frac{(-1)^k W_k}{p_3(\zeta)W}\right]V_k^{*(i+2)}(1)=0\qquad i=0,1 \tag{23}$$

gives the coefficients for a free-free beam:

$$C_1(\zeta)=\frac{1}{M(\zeta)}\left[b_2(1)a_{34}(0)-b_3(1)a_{24}(0)-b_4(1)a_{23}(0)\right]$$

$$C_2(\zeta)=\frac{1}{M(\zeta)}\left[c_1(1)a_{34}(0)+c_3(1)a_{14}(0)-c_4(1)a_{13}(0)\right]$$

$$C_3(\zeta)=\frac{1}{M(\zeta)}\left[c_1(1)a_{24}(0)-c_2(1)a_{14}(0)+c_4(1)a_{12}(0)\right]$$

$$C_4(\zeta)=\frac{1}{M(\zeta)}\left[c_2(1)a_{13}(0)-c_1(1)a_{23}(0)-c_3(1)a_{12}(0)\right] \tag{24}$$

For a clamped-free beam the boundary conditions are as follows:

$$\sum_{k=1}^{4} C_k V_k^{*i}(0) = 0\,, \quad \sum_{k=1}^{4}\left[C_k + \frac{(-1)^k W_k}{p_3(\zeta) W}\right] V_k^{*(i+2)}(1) = 0 \quad i = 0,\, 1 \tag{25}$$

and the coefficients C_k are:

$$C_1(\zeta) = \frac{1}{N(\zeta)}\left[c_3(1)d_{24}(0) - c_2(1)d_{34}(0) - c_4(1)d_{23}(0)\right]$$

$$C_2(\zeta) = \frac{1}{-N(\zeta)}\left[c_3(1)d_{14}(0) - c_1(1)d_{34}(0) - c_4(1)d_{13}(0)\right]$$

$$C_3(\zeta) = \frac{1}{N(\zeta)}\left[c_2(1)d_{14}(0) - c_1(1)d_{24}(0) - c_4(1)d_{12}(0)\right]$$

$$C_4(\zeta) = \frac{1}{-N(\zeta)}\left[c_2(1)d_{13}(0) - c_1(1)d_{23}(0) - c_3(1)d_{12}(0)\right] \tag{26}$$

Introduced functions and values occuring in (24), (26) are given as:

$$a_{ij} = V_i^{*''} V_j^{*'''} - V_i^{*'''} V_j^{*''}\,, \; d_{ij} = V_i^{*} V_j^{*'} - V_i^{*'} V_j^{*}\,,$$

$$b_i = B(\zeta) V_i^{*''} + A(\zeta) V_i^{*'''}\,, \; c_i = B(\zeta) V_i^{*''} - A(\zeta) V_i^{*'''}\,,$$

$$A(\zeta) = \frac{1}{W(\zeta)} \sum_{k=1}^{4} (-1)^{k+1} W_k(\zeta) V_k^{*''}(1)\,, \; B(\zeta) = \frac{1}{W(\zeta)} \sum_{k=1}^{4} (-1)^{k+1} W_k(\zeta) V_k^{*'''}(1)\,,$$

and denominators M, N:

$$M(\zeta) = p_3(\zeta)\big[a_{24}(0)a_{13}(1) + a_{24}(1)a_{13}(0) - a_{34}(0)a_{12}(1) - \\ - a_{34}(1)a_{12}(0) - a_{23}(0)a_{14}(1) - a_{24}(1)a_{14}(0)\big]$$

$$N(\zeta) = p_3(\zeta)\big[d_{34}(0)a_{12}(1) + d_{12}(0)a_{34}(1) - d_{24}(0)a_{13}(1) - \\ - d_{13}(0)a_{24}(1) + d_{23}(0)a_{14}(1) + d_{14}(0)a_{23}(1)\big]$$

Conclusions

The presented solution for the free vibration problem may be used for numerical calculations. The presented method may be used to analyze stepped non-uniform beams with additional discrete elements.

References

[1] Auciello N.M., Maurizi M.J., On the natural vibrations of tapered beams with attached inertia elements, Journal of Sound and Vibration 1997, 199(3), 522-530.

[2] Zhou D., Cheung Y.K., The free vibration of a type of tapered beams, Computer Methods in Applied Mechanics and Engineering 2000, 188(1-3).

[3] Elishakoff I., Becquet R., Closed-form solutions for natural frequency for inhomogeneous beams with one sliding support and the other pinned, Journal of Sound and Vibration 2000, 238(3), 529-539.

[4] Zamojska I., Analiza drgań własnych jednowymiarowych układów dyskretno-ciągłych. Zastosowanie metody funkcji Greena, praca doktorska, 2006.

[5] Kukla S., Dynamiczne funkcje Greena w analizie drgań własnych ciągłych i dyskretno-ciągłych układów mechanicznych, Monografie 64, Wydawnictwo Politechniki Częstochowskiej, Częstochowa 1999.

[6] Kukla S., Zamojska I., Application of the Green's function method in free vibration analysis of non-uniform beams, Scientific Research of the Institute of Mathematics and Computer Science 2005,1(4), 87-94.