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NUMERICAL SOLUTION OF FRACTIONAL EULER-LAGRANGE EQUATION WITH MULTIPOINT BOUNDARY CONDITIONS

Tomasz Blaszczyk[1], Mariusz Ciesielski[2]

[1]Institute of Mathematics, [2]Institute of Computer and Information Sciences
Czestochowa University of Technology, Poland
tomasz.blaszczyk@im.pcz.pl, mariusz.ciesielski@icis.pcz.pl

Abstract. In this paper we consider an ordinary fractional differential equation containing a composition of left and right fractional derivatives. This type of equation is known in literature as the fractional Euler-Lagrange equation. We considered this equation with multipoint boundary conditions. We proposed a numerical scheme using the finite difference method. In the final part of the paper, examples of the solutions are shown.

Introduction

In this study the fractional Euler-Lagrange equation (FELE) is considered. This type of equations is obtained when the minimum action principle and fractional integration by parts rule are applied. The fractional operator in this equation contains left and right derivatives simultaneously. It should be noted that many authors [1-7] elaborated some forms of the FELE.

Fractional differential equations appear naturally in a number of fields such as physics, mechanics, control theory, electrotechnics, bioengineering, finance theory and many other disciplines [8-10]. The important problem is how to find solutions of the FELE. Using fixed point theorems [5, 6], one can obtain analytical results represented by a series of alternately left and right fractional integrals and therefore it is difficult in any practical calculations. On the other hand, in references [11-13] we can find a numerical approach to the solution of ordinary differential equations with left and right fractional derivatives with the first kind of boundary conditions. In our work, we shall present numerical solutions of the FELE with a multipoint boundary condition.

1. Statement of the problem

We consider the following fractional differential equation of order $\alpha \in (0, 1)$ in time interval $t \in [0, b]$

$$^{C}D^{\alpha}_{b-} D^{\alpha}_{0+} f(t) + \lambda f(t) = 0 \tag{1}$$

where operators D^{α} are the left and right fractional derivatives in Riemann--Liouville (2) and Caputo (3) senses defined as [14]:

$$D^{\alpha}_{0+} f(t) = D\, I^{1-\alpha}_{0+} f(t) \tag{2}$$

$$^{C}D^{\alpha}_{b-} f(t) = -I^{1-\alpha}_{b-} D\, f(t) \tag{3}$$

and operators I^{α} are fractional integrals of order α defined in [14]:

$$\begin{aligned} I^{\alpha}_{0+} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, \quad \text{for } t>0 \\ I^{\alpha}_{b-} f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}} \mathrm{d}\tau, \quad \text{for } t<b \end{aligned} \tag{4}$$

The following relation between both definitions (2) and (3) takes place [14]:

$$D^{\alpha}_{0+} f(t) = {}^{C}D^{\alpha}_{0+} f(t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0) \tag{5}$$

Eq. (1) is supplemented by the multipoint boundary conditions

$$\begin{aligned} f(0) &= F_0 \\ f(a) &= F_a, \quad a \in (0,b] \end{aligned} \tag{6}$$

2. Numerical solution

Now we present numerical schemes for eq. (1). We introduce the homogenous grid of nodes

$$0 = t_0 < t_1 < t_2 < \ldots < t_i < t_{i+1} < \ldots < t_N = b, \qquad t_i = i\,\Delta t, \quad \Delta t = b/N \tag{7}$$

The value of function f at the moment of time t_i is denoted as $f_i = f(t_i)$.

Next, we determine the numerical schemes for both fractional operators occurring in eq. (1). The value of the left Riemman-Liouville derivative (2) (internal operator) at point t_i can be approximated as [11]

$$\begin{aligned} D^{\alpha}_{0+} f(t)\Big|_{t=t_i} &= f_0 \frac{t_i^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_i} \frac{f'(\tau)}{(t_i-\tau)^{\alpha}} \mathrm{d}\tau \\ &\cong f_0 \frac{t_i^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i-1} \frac{f_{j+1}-f_j}{\Delta t} \int_{t_j}^{t_{j+1}} \frac{1}{(t_i-\tau)^{\alpha}} \mathrm{d}\tau \end{aligned} \tag{8}$$

$$= f_0 \frac{t_i^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i-1} \frac{f_{j+1}-f_j}{\Delta t} \cdot \frac{(t_i-t_j)^{1-\alpha} - (t_i-t_{j+1})^{1-\alpha}}{1-\alpha} = (\Delta t)^{-\alpha} \sum_{j=0}^{i} f_j\, v(i,j)$$

where

$$v(i,j)=\frac{1}{\Gamma(2-\alpha)}\begin{cases}(1-\alpha)i^{-\alpha}+(i-1)^{1-\alpha}-i^{1-\alpha} & \text{for } j=0\\ (i-j+1)^{1-\alpha}-2(i-j)^{1-\alpha}+(i-j-1)^{1-\alpha} & \text{for } j=1,\dots,i-1\\ 1 & \text{for } j=i\end{cases} \quad (9)$$

Denoting by $g(t)=D^{\alpha}_{0+}f(t)$, we can find the discrete form of the composition of operators (2) and (3)

$$\begin{aligned}{}^{C}D^{\alpha}_{b-}D^{\alpha}_{a+}f(t)\Big|_{t=t_i} &= {}^{C}D^{\alpha}_{b-}g(t)\Big|_{t=t_i} = \frac{-1}{\Gamma(1-\alpha)}\int_{t_i}^{t_N}\frac{g'(\tau)}{(\tau-t_i)^{\alpha}}\mathrm{d}\tau\\ &\cong \frac{-1}{\Gamma(1-\alpha)}\sum_{j=i}^{N-1}\frac{g_{j+1}-g_j}{\Delta t}\int_{t_j}^{t_{j+1}}\frac{1}{(\tau-t_i)^{\alpha}}\mathrm{d}\tau\\ &= \frac{-1}{\Gamma(1-\alpha)}\sum_{j=i}^{N-1}\frac{g_{j+1}-g_j}{\Delta t}\cdot\frac{(t_{j+1}-t_i)^{1-\alpha}-(t_j-t_i)^{1-\alpha}}{1-\alpha}=(\Delta t)^{-\alpha}\sum_{j=i}^{N}g_j w(i,j)\end{aligned} \quad (10)$$

where

$$w(i,j)=\frac{1}{\Gamma(2-\alpha)}\begin{cases}1 & \text{for } j=i\\ (j-i+1)^{1-\alpha}-2(j-i)^{1-\alpha}+(j-i-1)^{1-\alpha} & \text{for } j=i+1,\dots,N-1\\ (N-i-1)^{1-\alpha}-(N-i)^{1-\alpha} & \text{for } j=N\end{cases} \quad (11)$$

Using formulas (8) and (10), we can describe a discrete form of the fractional operator in eq. (1)

$${}^{C}D^{\alpha}_{b-}D^{\alpha}_{0+}f(t)\Big|_{t=t_i} \cong (\Delta t)^{-2\alpha}\sum_{j=i}^{N}\left[w(i,j)\sum_{k=0}^{j}v(j,k)f_k\right] \quad (12)$$

Taking into account eq. (1) with multipoint boundary conditions (6), we have

$$\begin{aligned}&f_0=F_0\\ &(\Delta t)^{-2\alpha}\sum_{j=i}^{N}\left[w(i,j)\sum_{k=0}^{j}v(j,k)f_k\right]+\lambda f_i=0 \quad \text{for } i=1,\dots,N-1\\ &\left(1-\frac{a-t_m}{\Delta t}\right)f_m+\frac{a-t_m}{\Delta t}f_{m+1}=F_a\end{aligned} \quad (13)$$

where $m=\lceil a/\Delta t\rceil-1$ and $\lceil x\rceil$ denotes is the smallest integer not less than *x*. In order to obtain the numerical solution of eq. (1), we need to solve the system of algebraic equations (13). If point *a* does not overlap the node in grid (7) (i.e. the point a lies in time interval (t_m, t_{m+1}]), then we use linear interpolation which is determined by the third equation in (13).

3. Examples of computations

In this section we present the results of the calculations obtained by our numerical approach.

Example 1. In the example, eq. (1) with parameter $\lambda = 0$ is considered. The analytical solution of eq. (1) with boundary condition $f(b = 1) = 1$ is given in form $f(t) = t^{\alpha}$ [13]. Assuming the boundary condition as $f(a = 0.5) = 0.5^{\alpha}$ we realized the numerical calculations for $N = \{32, 64, 128, 256, 512, 1024\}$ and we determined f_N the values (i.e. for $t = b$). Table 1 presents the values of the numerical errors: $ERR = |(f(b) - f_N)|/f(b)$ for the various values of parameter α. It should be noted that the numerical errors decrease with an increasing discretization number N.

Example 2. Here we analysed the case of eq. (1) with parameter $\lambda = -10$. We simulated the influence of parameter α from the list, $\alpha = \{0.3, 0.5, 0.7\}$, on the solution. We divided time domain $t \in [0, 1]$ into $N = 1000$ subintervals. Figure 1 shows the solutions of eq. (1) with boundary conditions $f(1) = 1$ (left-side) and $f(0.5) = 1$ (right-side).

Table 1

Values of numerical errors *ERR*

	$\alpha = 0.001$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 0.999$
$N = 32$	$1.613\cdot10^{-5}$	$1.637\cdot10^{-3}$	$4.989\cdot10^{-3}$	$7.928\cdot10^{-3}$	$8.865\cdot10^{-3}$	$4.928\cdot10^{-3}$	$6.132\cdot10^{-5}$
$N = 64$	$7.938\cdot10^{-6}$	$8.006\cdot10^{-4}$	$2.432\cdot10^{-3}$	$3.936\cdot10^{-3}$	$4.619\cdot10^{-3}$	$2.768\cdot10^{-3}$	$3.599\cdot10^{-5}$
$N = 128$	$3.937\cdot10^{-6}$	$3.957\cdot10^{-4}$	$1.198\cdot10^{-3}$	$1.961\cdot10^{-3}$	$2.386\cdot10^{-3}$	$1.526\cdot10^{-3}$	$2.067\cdot10^{-5}$
$N = 256$	$1.961\cdot10^{-6}$	$1.967\cdot10^{-4}$	$5.937\cdot10^{-4}$	$9.785\cdot10^{-4}$	$1.225\cdot10^{-3}$	$8.292\cdot10^{-4}$	$1.168\cdot10^{-5}$
$N = 512$	$9.785\cdot10^{-7}$	$9.802\cdot10^{-5}$	$2.953\cdot10^{-4}$	$4.888\cdot10^{-4}$	$6.253\cdot10^{-4}$	$4.455\cdot10^{-4}$	$6.509\cdot10^{-6}$
$N = 1024$	$4.888\cdot10^{-7}$	$4.892\cdot10^{-5}$	$1.472\cdot10^{-4}$	$2.443\cdot10^{-4}$	$3.180\cdot10^{-4}$	$2.372\cdot10^{-4}$	$3.590\cdot10^{-6}$

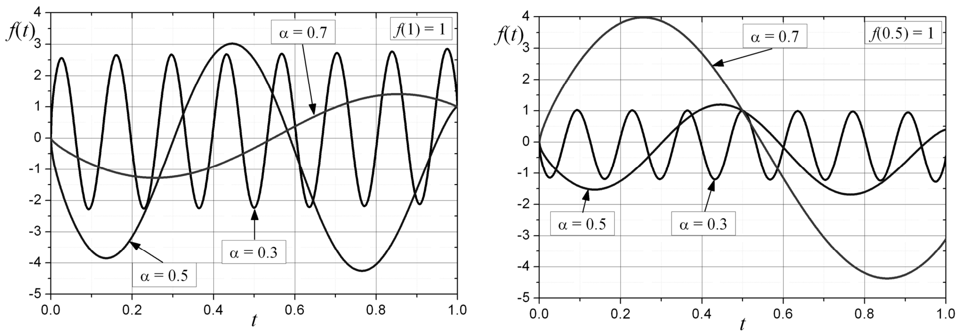

Fig. 1. Solutions of eq. (1) with $\lambda = -10$ and boundary conditions $f(1) = 1$ (left-side) and $f(0.5) = 1$ (right-side) for different values of parameter α

Example 3. The last example is similar to example 2. We assumed the following parameters: $\alpha = 0.5$, $\lambda = -10$, $t \in [0, 1]$, $N = 1000$. Here we determined the influence of the changes of boundary conditions on the solution. Figure 2 presents the solutions of eq. (1) with boundary conditions: $f(1) = \{0.5, 1.0, 1.5\}$ (left-side) and $f(0.5) = \{0.5, 1.0, 1.5\}$ (right-side).

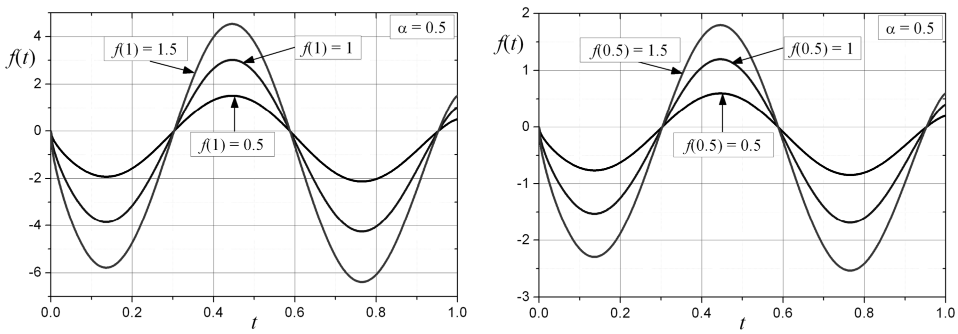

Fig. 2. Solutions of eq. (1) with $\lambda = -10$, $\alpha = 0.5$ and variables values of boundary conditions at $f(1)$ (left-side) and $f(0.5)$ (right-side)

Conclusions

In summary we proposed the FDM for the FELE with multipoint boundary conditions. We obtained the FDM scheme which includes one of the boundary conditions inside of the considered time domain. This approach offers new possibilities in physical processes modelling. Analysing the plots in this work, we observed the occurrence of oscillation for $\lambda < 0$. We also noted that the numerical solutions for case $\lambda = 0$ are convergent to the analytical one.

References

[1] Agrawal O.P., Formulation of Euler-Lagrange equations for fractional variational problems, J. Math. Anal. Appl. 2002, 272, 368-379.

[2] Agrawal O.P., Generalized variational problems and Euler-Lagrange equations, Comput. Math. Appl. 2010, 59, 1852-1864.

[3] Baleanu D., Trujillo D.J.J., On exact solutions of a class of fractional Euler-Lagrange equations, Nonlinear Dyn. 2008, 52, 331-335.

[4] Klimek M., Lagrangean and Hamiltonian fractional sequential mechanics, Czech. J. Phys. 2002, 52, 1247-1253.

[5] Klimek M., G-Meijer functions series as solutions for certain fractional variational problem on a finite time interval, Journal Europeen des Systemes Automatises (JESA) 2008, 42, 653-664.

[6] Klimek M., On Solutions of Linear Fractional Differential Equations of a Variational Type, The Publishing Office of the Czestochowa University of Technology, Czestochowa 2009.

[7] Riewe F., Nonconservative Lagrangian and Hamiltonian mechanics, Phys. Rev. E 1996, 53, 1890-1899.

[8] Hilfer R., Applications of Fractional Calculus in Physics, World Scientific, Singapore 2000.

[9] Kilbas A.A., Srivastava H.M., Trujillo J.J., Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam 2006.

[10] Leszczynski J.S., Blaszczyk T., Modeling the transition between stable and unstable operation while emptying a silo, Granular Matter 2011, 13, 429-438.

[11] Sommacal L., Melchior P., Cabelguen J.M., Ostaloup A., Ijepeert A., Fractional multimodels of the gastrocnemius muscle for tetanus pattern, [in:] J. Sabatier, O.P. Agrawal, J.A.T. Machado, Advances in Fractional Calculus. Theoretical Developments and Applications in Physics and Engineering, Springer-Verlag, 2007, 271-285.

[12] Blaszczyk T., Ciesielski M., Fractional Euler-Lagrange equations-numerical solutions and applications of reflection operator, Scientific Research of the Institute of Mathematics and Computer Science 2010, 2(9), 17-24.

[13] Blaszczyk T., Application of the Rayleigh-Ritz method for solving fractional oscillator equation, Scientific Research of the Institute of Mathematics and Computer Science 2009, 2(8), 29-36.

[14] Blaszczyk T., Ciesielski M., Klimek M., Leszczynski J., Numerical solution of fractional oscillator equation, Applied Mathematics and Computation 2011, 218, 2480-2488.

[15] Oldham K.B., Spanier J., The fractional calculus. Theory and applications of differentiation and integration to arbitrary order, Academic Press, New York 1974.