Scientific Research of the Institute of Mathematics and Computer Science, 2(10) 2011, 135-142

BALANCING METHOD OF UNSUSTAINABLE PRODUCTION AND CONSUMPTION MODEL

Marek Ładyga, Maciej Tkacz

Institute of Mathematics, Czestochowa University of Technology, Poland
marek.ladyga@im.pcz.pl, maciej.tkacz@im.pcz.pl

Abstract. The paper presents an iterative method to bring about a sustainable production - - consumption model. It is assumed that the aggregate demand exceeds the aggregate supply at a given fixed time period. The proposed method consists in the fact that in each iterative step and each contractor actually makes a concession. This concession does not exceed the given limits of their capacity for production or consumption. In each subsequent iteration step, the difference between the demand and supply is less than in the previous one. To achieve balance in the model, the proposed method needs a finite number of steps. This number does not exceed the number of contractors.

1. Model description

In [1], the authors define a model of unsustainable production and consumption, in which there are $n \geq 2$ contractors, with $m \geq 1$ producers and $n - m \geq 1$ consumers of the goods. Let the n-dimensional vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ represent the initial values of the supply and demand. Coefficients p_i, $i \in \{1, \ldots, m\}$ represent the values of supply of the m producers whilst the remaining ones represent the demands of the consumers. For convenience, the demands will be denoted by negative values.

It is assumed that the aggregated demand exceeds the aggregated supply in a considered time period. This fact can be expressed by the following conditions

1. $\bigwedge_{i \in N} i \leq m \Rightarrow p_i \rangle\, 0$

2. $\bigwedge_{i \in N} m + 1 \leq i \leq n \Rightarrow p_i \langle\, 0$

3. $\sum_{i=1}^{n} p_i \,\langle\, 0$

Vector **p** meeting the above conditions is called an unadjusted vector. Condition 3 reflects the fact that the production-consumption model is unsustainable. Therefore, there is a natural problem of the sustainability of such a model.

Denote a K set of customers (size of K set is equal to n), and a KR set of criteria (size of KR set is equal to k) on the basis of which can lead to a sustainable model. To determine the actual concessions to contractors, we introduce mapping coefficients.

Definition 1. The projection of $f: K \times KR \to R_+ \cup \{0\}$ is called a mapping projection $\Leftrightarrow$ if for every $i \in N_n$ an $j \in N_k$ appears, where $N_s = \{1, 2, \ldots, s\}$.

It is convenient to write the value of mapping f in the form of an F matrix

$$F = \begin{bmatrix} f(1,\ 1) & f(1,\ 2) & \ldots & f(1,\ k) \\ f(2,\ 1) & f(2,\ 2) & \ldots & f(2,\ k) \\ \vdots & & & \vdots \\ f(n,\ 1) & f(n,\ 2) & \ldots & f(n,\ k) \end{bmatrix}$$

Element $f(i,\ j)$ of matrix F informs how i-contractor meets j-criteria. It is obvious that not every criterion is equally important. It is therefore expedient to introduce vector weights of the criteria. Denote this vector by $w = (w_1, w_2, \ldots, w_k)$, where for every $i \in N_k, w_i \rangle 0$ and $w_1 + w_2 + \ldots + w_k = 1$.

Definition 2. An A matrix is called the matrix associated with the mapping of correcting f, which is written as

$$A(f) = [a_{ij}], i \in N_n, j \in N_k \text{ if}$$

$$a_{ij} = w_j \cdot f(i,\ j) \cdot \left(\sum_{t=1}^{n} f(t,\ j) \right)^{-1}$$

The elements of matrix 100 $A(f)$ inform about the behavior of the contractors to the various criteria in terms of their weight. Aggregation of the *i*-line allows one to assess the relationship of the i-contractor to all the criteria. Crucial to determining the final concessions of contractors is mapping vector f determined by $a = (a_1, \ldots, a_n) = A(f) \cdot e_k^T$ where $e_k = \underbrace{(1, \ldots, 1)}_{k}$.

2. Description of iterative algorithm

Let us take the following vectors:

a) unadjusted vector of demand-supply $p = (p_1, \ldots, p_n)$

b) vector of maximum concessions $u = (u_1, \ldots, u_n)$ such that $p_1 + \ldots + p_n + u_1 + \ldots + u_n \geq 0, u_i \geq 0, i \in N_n$

c) vector $a = (a_1, \ldots, a_n)$ associated with mapping f.

Assume that $p^o = p$, $a^o = a$, $N^o = N_n$. Then for $s \geq 1$ denoting the *s*-iteration can be defined

$$r^s = -\sum_{i=1}^{n} p_i^{s-1} \tag{1}$$

$$N^s = \left\{ i \in N^o : \sum_{j \leq s} a_i^{j-1} \cdot r^j \leq u_i \right\} \tag{2}$$

$$a^s = \left(a_1^s, a_2^s, \ldots, a_n^s \right)$$

where

$$a_i^s = \begin{cases} 0 & \text{when} \quad i \notin N^s \\ a_i^{s-1} \cdot \left(B^{s-1} \right)^{-1} & \text{when} \quad i \in N^s \end{cases} \tag{3}$$

$$B^{s-1} = \sum_{j \in N^{s-1}} a_j^{s-1}$$

$$\overline{N}^s = N^{s-1} \setminus N^s \tag{4}$$

$$p^s = \left(p_1^s, \ldots, p_n^s \right)$$

where

$$p_i^s = \begin{cases} p_i^{s-1} + r^s a_i^{s-1} & \text{when} \quad i \in N^s \\ p_i + u_i & \text{when} \quad i \in \overline{N}^s \\ p_i^{s-1} & \text{when} \quad i \in \bigcup_{j < s} \overline{N}^j \end{cases} \tag{5}$$

This iterative process is finished for s such that $r^s = 0$. Below we give an economic interpretation of the size (1-5). Number (1) means the difference which remains after balancing the model after $s-1$ iterations.

Vector a^s sets the price contractors' concessions against the criteria in the *s*-iteration, the *i*-coordinate of vector $r^s a^{s-1}$ determines the amount by which it is proposed to verify the supply or demand to the *i*-contractor at an *s*-iteration.

During verification, one must be careful not to exceed the maximum concessions of the *i*-contractor, i.e. $\sum_{j \leq s} a_i^{j-1} r^j \leq u_i$. Therefore, sets N^s and $\overline{N}^s$ were introduced. The first of these is the set of contractors, for which the proposed concession, included after the *s*-th iteration, does not exceed their maximum concessions. Set $\overline{N}^s$ is a collection of contractors for which the total exceeded the limit of concession opportunities in the s-iteration. Model (5) sets the reduced supply and demand in the s-th iteration.

When defining a p^s vector, the principle of division into three sets of customers is being used:

a) A set of contractors, who as a result of verification of supply and demand in previous iterations agreed for the demand or supply to limit their opportunities. The demand or supply of these contractors left the s-th iteration unchanged $\left(p_i^{s-1} = p_i^s\right)$.

b) A set of contractors, who as a result of the proposed review of the demand or supply, the s-th iteration exceeds the limit of capabilities of their concessions. The demand or supply of these contractors is set as the limit of their capabilities $\left(p_i^s = p_i + u_i\right)$.

c) A set of contractors, who as a result of the proposed changes in demand or supply in the s-th iteration still have reserve concessions. The demand or supply of these contractors calculated in the previous iteration varies by size $r^s a_i^{s-1}$, $\left(p_i^s = p_i^{s-1} + r^s a^{s-1}\right)$.

Vector which is a vector of the s-th iteration of the balancing model is called the i-correction unadjusted vector. A vector which balances the model is called the total correction unadjusted vector. The total size of all the concessions in the contractors in this s-iteration is called the local amendment row s. The amount by which the model's imbalance was reduced from the first to the s-th iteration is called a global amendment of the unadjusted s vector row. Interpretations of these can be contained in the following definition.

Definition 3. Number $S^k = \sum_{i \in N^k} r^k a_i^{k-1} + \sum_{i \in \overline{N}^k} \left(u_i - \sum_{j=1}^{k-1} r^i a_i^{j-1} \right)$ is called the local amendment k row for unadjusted p vector.

Definition 4. Number $\overline{S}^k = \sum_{j \le k} S^j$ is called the global amendment k row for unadjusted p vector.

3. Certain features of the algorithm

In this section some formal results connected with the algorithm as well as some practical interpretations of the above definitions will be presented.

Let $N(s)=\{s\in N:\ r^s\neq 0\}$ denote the set of iteration indicators, for which the p^s vector is not the total correction of the p vector, $s_o=\min\limits_s (r^s=0)$ means the iteration in which as a result the vector of total correction of the unadjusted vector is received, $N(s,o)=N(s)\cup\{s_o\}$.

Lemma 1. For every s belonging to a set of $N(s,o)$, there is $N^o=\overline{N}^1\cup\overline{N}^2\cup\ldots\cup\overline{N}^s\cup N^s$, when $\overline{N}^i\cap\overline{N}^j=\varnothing$ for $i\neq j\leq s$ and $N^s\cap\overline{N}^i=\varnothing$ for $i\langle\ s$.

The proof of the lemma is on the induction run due to s.

a) $s=1$, from formula (4), it shows that $\overline{N}^1=N^o\setminus N^1$, therefore $N^1\cap\overline{N}^1=\varnothing$ and $N^oN^1\cup\overline{N}^1$, which ends the proof of the lemma for $s=1$.

b) It is assumed that the lemma states the thru for $s\leq k$, so $N^o=\overline{N}^1\cup\overline{N}^2\cup\ldots\cup\overline{N}^k\cup N^k$ and $\overline{N}^i\cap\overline{N}^j=\varnothing$ for $i\neq j\leq k$ and $\overline{N}^i\cap N^k=\varnothing$ for $i\leq k$.

c) Let $s=k+1$. From formula (4), it shows that $N^k=N^{k+1}\cup\overline{N}^{k+1}$, therefore $N^{k+1}\subset N^k$ i $\overline{N}^{k+1}\subset N^k$.

From the hypothesis of induction and the above inclusions, we have $N^o=\overline{N}^1\cup\overline{N}^2\cup\ldots\cup\overline{N}^k\cup N^k=\overline{N}^1\cup\overline{N}^2\cup\ldots\overline{N}^k\cup\overline{N}^{k+1}\cup N^{k+1}$ and $N^{k+1}\cap\overline{N}^{\,i}=\varnothing$, $\overline{N}^{k+1}\cap\overline{N}^i=\varnothing$ for $i\leq k$ which ends the proof.

Statement 1. For every k belonging to the set of $N(s,o)$, there is equality $\left(p^k-p^{k-1},e_n\right)=S^k$.

Proof. Using lemma 1.

$$\left(p^k-p^{k-1},e_n\right)=\sum_{i\in N^o}\left(p_i^k-p_i^{k-1}\right)=\sum_{i\in N^k}\left(p_i^k-p_i^{k-1}\right)+\sum_{i\in\overline{N}^k}\left(p_i^k-p_i^{k-1}\right)+\sum_{\substack{i\in\cup\overline{N}^j\\ j\langle k}}\left(p_i^k-p_i^{k-1}\right)=$$

$$=\sum_{i\in N^k} r^k a_i^{k-1}+\sum_{i\in\overline{N}^k}\left(p_i^o+u_i-p_i^{k-1}\right)$$

Formula (4) shows that for every k belonging to the set of $N(s,o)$, there is equality $N^{k-1}=N^k\cup\overline{N}^k$, hence $N^1\supset\ldots\supset N^k\supset\overline{N}^k$ therefore, for every $i\langle\ k$ there is inclusion $\overline{N}^k\subset N^i$.

Formula (5) shows that for every *i* belonging to a set of N^m, there is equality $p_i^m - p_i^{m-1} = r^m a_i^{m-1}$, therefore, for every $i \in \overline{N}^k$ there is:

$$p_i^o + u_i - p_i^{k-1} = u_i + \left(p_i^o - p_i^1\right) + \left(p_i^1 - p_i^2\right) + \ldots + \left(p_i^{k-2} - p_i^{k-1}\right) =$$
$$= u_i - r^1 a_i^o - r^2 a_i^1 - \ldots - r^{k-1} a_i^{k-2} = u_i - \sum_{j \le k-1} r^j a_i^{j-1}$$

Then $\left(p^k - p^{k-1}, e_n\right) = \sum_{i \in N^k} r^k a_i^{k-1} + \sum_{i \in \overline{N}^k} \left(u_i - \sum_{j=1}^{k-1} r^j a_i^{j-1} \right) = S^k$, which ends the statement proof.

Statement 2. For every k belonging to the set of $N(s,o)$, there is equality $\overline{S}^k = \left(p^k - p^o, e_n\right)$.

Proof. From statement 1 we have

$$\left(p^k - p^o, e_n\right) = \sum_{i=1}^{n} \left(p_i^k - p_i^o\right) = \sum_{i=1}^{n} \left(\left(p_i^k - p_i^{k-1}\right) + \left(p_i^{k-1} - p_i^{k-2}\right) + \ldots + \left(p_i^1 - p_i^o\right)\right) =$$
$$= \sum_{i=1}^{n} \left(p_i^k - p_i^{k-1}\right) + \ldots + \sum_{i=1}^{n} \left(p_i^1 - p_i^o\right) = \left(p^k - p^{k-1}, e_n\right) + \ldots + \left(p^1 - p^o, e_n\right) = S^k + \ldots + S^1 =$$
$$= \sum_{j=1}^{k} S^j = \overline{S}^k, \text{ which ends the proof.}$$

Statement 3. The following conditions are equivalent:

a) $\left(p^s, e_n\right) = 0$,
b) $\overline{N}^s = \emptyset$,
c) $\left(r^{s+1} a^s, e_n\right) = 0$,
d) $\overline{S}^s = r^1$,
e) $N^s = N^{s-1}$,
f) $S^s = -\left(p^{s-1}, e_n\right)$.

Proof. The equalities from Figure 1 will be proved.

$(a) \Leftrightarrow (b)$

$\left(p^s, e_n\right) = \sum_{i \in \overline{N}^s} c_i$ when $c_i = u_i + p_i^o - p_i^{s-1} - r^s a_i^{s-1}$, but for every $i \in \overline{N}^s$, there is $c_i \rangle\ 0$. Then $\left(p^s, e_n\right) = 0$ then and only if $\overline{N}^s = \emptyset$.

$(a) \Leftrightarrow (c)$

$$
\begin{array}{ccccc}
(b) & \Leftrightarrow & (a) & \Leftrightarrow & (f) \\
\Updownarrow & & \Updownarrow & & \\
(e) & & (c) & & \\
 & & \Updownarrow & & \\
 & & (d) & &
\end{array}
$$

Fig. 1. Proof outline

From formulas (1) and (3) and the scalar ratio properties, it follows that

$$\left(r^{s+1}\cdot a^s, e_n\right)=r^{s+1}\left(a^s, e_n\right)=r^{s+1}=-\left(p^s, e\right)$$

$(c)\Leftrightarrow(d)$ and using formula (1) we have

$$\overline{S}^s+\left(r^{s+1}\cdot a^s, e_n\right)=\left(p^s-p^o, e_n\right)+\left(r^{s+1}a^s, e_n\right)=\left(p^s, e_n\right)-\left(p_o, e_n\right)+r^{s+1}\left(a^s, e_n\right)=$$
$$=-r^{s+1}+r^1+r^{s+1}=r^1$$

Therefore $\overline{S}^s = r^1$ then and only if $\left(r^{s+1}a^s, e_n\right)=0.$

$(e)\Leftrightarrow(b)$

From formula (4) it follows that $N^{s-1}=N^s\cup\overline{N}^s$ and $N^s\cap\overline{N}^s=\varnothing$. Therefore $N^{s-1}=N^s$ then and only if $\overline{N}^s=\varnothing$.

$(f)\Leftrightarrow(a)$

From statement 1 it follows that $S^s=\left(p^s-p^{s-1}, e_n\right)=\left(p^s, e_n\right)-\left(p^{s-1}, e_n\right)$. Therefore $S^s=-\left(p^{s-1}, e_n\right)$ if and only if $\left(p^s, e_n\right)=0$, which ends the proof.

Each of the accounts $(a)-(f)$ is equivalent to the $r^{s+1}=0$ condition, so each of these conditions inform about finding the vector balancing the unbalanced production and consumption model. This iteration is done in a finite number of steps. The number of steps does not exceed the number of contractors.

References

[1] Dittmann P., Dittmann I., Szabela-Pasierbińska E., Szpulak A., Prognozowanie w zarządzaniu przedsiębiorstwem, Oficyna a Wolters Kluwer Business, Kraków 2009.

[2] Guzik B., Podstawowe modele w badaniu efektywności gospodarczej i społecznej, Wydawnictwo Uniwersytetu Ekonomicznego, Poznań 2009.

[3] Pawełek B., Metody normalizacji zmiennych w badaniach porównawczych złożonych, zjawisk ekonomicznych, Uniwersytet Ekonomiczny, Kraków 2008.

[4] Trzaskalik T. (ed.), Metody wielokryterialne na polskim rynku finansowym, PWE, Warszawa 2006, in Polish

[5] Ładyga M., Tkacz M., The unsustainable production and consumption model, Polish Journal of Management Studies, Czestochowa University of Technology, Częstochowa 2011.