Journal of Applied Mathematics and Computational Mechanics 2014, 13(1), 61-73

APPLICATION OF A Z-TRANSFORMS METHOD FOR INVESTIGATION OF MARKOV G-NETWORKS

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Abstract. The purpose of this research paper is to find the expected incomes in open Markov queueing networks with incomes, positive and negative messages at any time by the multidimensional transformations. Investigations were carried out in cases when incomes from the network transitions between the states are deterministic functions not dependent on network states and time. It is assumed that all network systems are one-line. It was proved the theorem on the expression for the multidimensional *z*-transform. An algorithm was proposed for calculation of expected incomes. It is calculated an example on the PC.

Keywords: *open G-network, positive and negative messages, expected incomes, z-transforms method*

Introduction

Markov networks with incomes (HM-network) and G-networks are widely used as mathematical models of various objects in computer sciences and other fields [1-3]. For example, G-network can be used to model the behavior of viruses in the information and telecommunication systems and networks for forecasting expenditure when it is in contact with viruses [4, 5]. In this case an important task is to find such expenditures in the systems of the considered network. Previously in work [6] for finding the expected incomes of systems of HM-network with many-type messages networks a method was proposed of successive approximations, combined with the method of series. In it was considered a closed exponential HM-network with limited waiting times in queues queueing messages. In [7], it is proposed a method of finding the expected incomes of systems of HM-open queueing network (QN) with a one-line systems of QN (QS), positive and negative messages in the case, when the incomes from the state transitions are deterministic network functions depending on network states does not depend on time. Such a technique to find the expected network incomes of systems via PC for an infinite number of network states in a reasonable CPU time.

In this paper, we consider another method based on the application of multivariate z-transforms. If we introduce into consideration multidimensional z-transforms for expected network incomes of systems, we obtain for their relations in this way, as in [1, 8]. On the basis of these relations we can propose an algorithm for calculating the expected incomes [9, 10] for the open G-network with incomes.

1. QN description. Formulation of the problem

Consider an open QN G-network with n one-line QS. In QN S_i come from outside independent Poisson flows of positive and negative messages with intensities λ_{0i}^{+} and λ_{0i}^{-} respectively, $i=\overline{1,n}$. The service time of positive messages in the QS S_i is distributed exponentially with parameter μ_i, $i=\overline{1,n}$. The actions of negative messages are described in [2-5]. Positive message serviced in the QS S_i, with probability p_{ij}^{+} is directed to QS S_j as positive message, and with probability p_{ij}^{-} - as negative message, and with probability $p_{i0}=1-\sum_{j=1}^{n}\left(p_{ij}^{+}+p_{ij}^{-}\right)$ *message leaves the network to the external environment* (QS S_0), $i,j=\overline{1,n}$. The message during the transition from one to another QS brings its system, and some income of the first system is reduced by this amount respectively. Consider the case when the income from the state transitions are deterministic functions of the network, depending on network states. The network state will be the vector $k(t)=(k,t)=(k_1(t),k_2(t),\dots,k_n(t))$, where $k_i(t)$ - the number of messages at the time moment t in system S_i, $i=\overline{1,n}$.

Let $v_i(k,t)$ - the total expected income, which gets the system S_i during time t, if at the initial moment the network is in a state k, and assume that this function is differentiable in t. We introduce the following notations: I_i - a vector of dimension n, consisting of zeros, except for the component with number of i, which is equal to 1, $i=\overline{1,n}$; I_0 - zero n-vector; $u(x)=\begin{cases}1, x>0\\0, x\le 0\end{cases}$ - Heaviside function; $r_i(k)$ - income system S_i at a time when the network is in a state k. This means that, for example, if the network is in a state k at moment time $t+\Delta t$, then the expected income of system S_i will be $r_i(k)\Delta t$ during Δt plus the expected income $v_i(k,t)$, that the system has received in the previous t time units. Let $r_{0i}(k+I_i)$ - income of system S_i, when network makes a transition from state (k,t) in state $(k+I_i,t+\Delta t)$ during Δt; $-R_{i0}(k-I_i)$ - income of its system, if network makes a transition from state (k,t) in state $(k-I_i,t+\Delta t)$; $r_{ij}(k+I_i-I_j)$ - the income of system S_i, when the network makes its transition from state (k,t) to $(k+I_i-I_j,t+\Delta t)$ during Δt; $-r_{ij}(k-I_i-I_j)$ - income of QS S_i, when the network changes transition from state (k,t) to $(k-I_i-I_j,t+\Delta t)$ during time Δt, $i,j=\overline{1,n}$.

It has been shown that the system of differential equations for the income $v_i(k,t)$ has the form [4, 5]:

$$\frac{dv_i(k,t)}{dt}=\alpha_i(k)-\sum_{j=1}^{n}\left[\lambda_{0j}^{+}+\left(\lambda_{0j}^{-}+\mu_j\right)u(k_j)\right]v_i(k,t)+$$
$$+\sum_{j=1}^{n}\left(\lambda_{0j}^{+}v_i(k+I_j,t)+\left[\mu_j p_{j0}u(k_j)+\lambda_{0j}^{-}u(k_j)+\mu_j\sum_{\substack{c=1\\c\neq j}}^{n}p_{jc}^{-}(1-u(k_c))\right]v_i(k-I_j,t)\right)+$$
$$+\lambda_{0i}^{+}v_i(k+I_i,t)+\left[\mu_i p_{i0}u(k_i)+\lambda_{0i}^{-}u(k_i)+\mu_i\sum_{\substack{c=1\\c\neq i}}^{n}p_{ic}^{-}(1-u(k_c))\right]v_i(k-I_i,t)+ \quad (1)$$
$$+\sum_{\substack{j=1\\j\neq i}}^{n}\left[\mu_j p_{ji}^{+}u(k_j)v_i(k+I_i-I_j,t)+\mu_i p_{ij}^{+}u(k_i)v_i(k-I_i+I_j,t)+\right.$$
$$\left.+\mu_i p_{ij}^{-}v_i(k-I_i-I_j,t)\right]+\sum_{\substack{c,s=1\\c,s\neq i,c\neq s}}^{n}\left[\mu_s p_{sc}^{+}u(k_s)v_i(k+I_c-I_s,t)+\mu_c p_{cs}^{-}v_i(k-I_c-I_s,t)\right],$$

where

$$\alpha_i(k)=r_i(k)+\lambda_{0i}^{+}r_{0i}(k+I_i)-$$
$$-\left[\mu_i p_{i0}u(k_i)+\lambda_{0i}^{-}u(k_i)+\mu_i\sum_{\substack{c=1\\c\neq i}}^{n}p_{ic}^{-}(1-u(k_c))\right]R_{i0}(k-I_i)+$$
$$+\sum_{\substack{j=1\\j\neq i}}^{n}\left[\mu_j p_{ji}^{+}u(k_j)r_{ji}(k+I_i-I_j)-\mu_i p_{ij}^{+}u(k_i)r_{ij}(k-I_i+I_j)-\right. \quad (2)$$
$$\left.-\mu_i p_{ij}^{-}r_{ij}(k-I_i-I_j)\right],\ i=\overline{1,n}.$$

It is necessary to find the expected network incomes of systems during t, if we know its state at the initial time.

2. Analysis of incomes in network by *z*-transforms method

We introduce multidimensional z-tranforms for expected income of system S_i:

$$\varphi_i(z,t)=\sum_{\substack{k_i=0,\\ i=\overline{1,n}}}^{\infty} v_i(k,t)\prod_{l=1}^{n} z_l^{k_l},\ z\in\left\{(z_1,z_2,...,z_n)/\left|z_i\right|<1,\ i=\overline{1,n}\right\}.$$

Multiply the system of equations (2) to $\prod_{l=1}^{n} z_l^{k_l}$ and summing over all k_l, $l=\overline{1,n}$, from 0 to ∞. Consider some sums included into converted system of equations. It can be shown that

$$\sum\nolimits_{1j}(z,t)=\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}\sum_{j=1}^{n}\left[\lambda_{0j}^{+}+\left(\lambda_{0j}^{-}+\mu_j\right)u(k_j)\right]v_i(k,t)z_1^{k_1}z_2^{k_2}\cdot\ldots\cdot z_n^{k_n}=$$
$$=\sum_{j=1}^{n}\left[\lambda_{0j}^{+}+\lambda_{0j}^{-}+\mu_j\right]\left(\varphi_i(z,t)-\varphi_i^{\{j\}}(z,t)\right),\ j=\overline{1,n}, \tag{3}$$

where

$$\varphi_i^{\{j\}}(z,t)=\sum_{k_1=0}^{\infty}\ldots\sum_{k_{j-1}=0}^{\infty}\sum_{k_{j+1}=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}v_i\left(k_1,\ldots,k_{j-1},0,k_{j+1},\ldots k_n,t\right)z_1^{k_1}\cdot\ldots\cdot z_{j-1}^{k_{j-1}}z_{j+1}^{k_{j+1}}\cdot\ldots\cdot z_n^{k_n}=$$
$$=\sum_{\substack{k_l=0,\\ l=\overline{1,n},\, l\neq j}}^{\infty} v_i(k,t)\big|_{k_j=0}\prod_{\substack{l=1,\\ l\neq j}}^{n} z_l^{k_l},\ j=\overline{1,n}. \tag{4}$$

$$\sum\nolimits_{2j}(z,t)=\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}\sum_{j=1}^{n}\left[\mu_j p_{j0}u(k_j)+\lambda_{0j}^{-}u(k_j)+\mu_j\sum_{\substack{c=1\\ c\neq j}}^{n}p_{jc}^{-}\left(1-u(k_c)\right)\right]v_i\left(k-I_j,t\right)\prod_{l=1}^{n}z_l^{k_l}=$$
$$=\sum_{j=1}^{n}\left[\mu_j p_{j0}+\lambda_{0j}^{-}\right]z_j\,\varphi_i(z,t),\ j=\overline{1,n}. \tag{5}$$

$$\sum\nolimits_{3j}(z,t)=\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}\sum_{j=1}^{n}\lambda_{0j}^{+}v_i\left(k+I_j,t\right)\prod_{l=1}^{n}z_l^{k_l}=\sum_{j=1}^{n}\frac{\lambda_{0j}^{+}}{z_j}\left[\varphi_i(z,t)-\varphi_i^{\{j\}}(z,t)\right],\ i=\overline{1,n}. \tag{6}$$

$$\sum\nolimits_{4j}(z,t)=\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\mu_j\,p_{ji}^{+}\,u(k_j)v_i\left(k+I_i-I_j,t\right)\right]v_i\left(k+I_i-I_j,t\right)\prod_{l=1}^{n}z_l^{k_l}=$$
$$=\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_j\,p_{ji}^{+}\frac{z_j}{z_i}\left[\varphi_i(z,t)-\varphi_i^{\{i\}}(z,t)\right],\ i=\overline{1,n}. \tag{7}$$

$$\sum\nolimits_{5j}(z,t)=\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_i p_{ij}^{+}u(k_i)v_i(k+I_j-I_i,t)\prod_{l=1}^{n}z_l^{k_l}=$$
$$=\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_i p_{ij}^{+}\frac{z_i}{z_j}\left[\varphi_i(z,t)-\varphi_i^{\{j\}}(z,t)\right]i,\, j=\overline{1,n}. \qquad (8)$$

$$\sum\nolimits_{6s}(z,t)=\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}\sum_{\substack{c,s=1\\ c,s\neq i,c\neq s}}^{n}\mu_s p_{sc}^{+}u(k_s)v_s(k+I_c-I_s,t)\prod_{l=1}^{n}z_l^{k_l}=$$
$$=\sum_{\substack{c,s=1\\ c,s\neq i,c\neq s}}^{n}\mu_s p_{sc}^{+}\frac{z_s}{z_c}\left[\varphi_c(z,t)-\varphi_c^{\{c\}}(z,t)\right],\; c,s=\overline{1,n},\; c,s\neq i. \qquad (9)$$

$$\sum\nolimits_{7j}(z,t)=\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_i p_{ij}^{-}v_i(k-I_i-I_j,t)\prod_{l=1}^{n}z_l^{k_l}=$$
$$=\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_i p_{ij}^{-}\left[\varphi_i(z,t)-\varphi_i^{\{i\}}(z,t)-\varphi_i^{\{j\}}(z,t)\right],\; j=\overline{1,n}. \qquad (10)$$

$$\sum\nolimits_{8j}(z,t)=\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}\sum_{\substack{c,s=1\\ c,s\neq i,c\neq s}}^{n}\mu_s p_{sc}^{+}v_i(k-I_c-I_s,t)\prod_{l=1}^{n}z_l^{k_l}=$$
$$=\sum_{\substack{c,s=1\\ c,s\neq i,c\neq s}}^{n}\mu_s p_{sc}^{+}\left[\varphi_c(z,t)-\varphi_c^{\{c\}}(z,t)-\varphi_c^{\{s\}}(z,t)\right],\; c,s=\overline{1,n},\; c,s\neq i. \qquad (11)$$

Thus, using (3)-(10), the system of equations (1) after the above transformations we got approval.

Theorem. *The function* $\varphi_i(z,t)$ *satisfies the relation*

$$\frac{\partial\varphi_i(z,t)}{\partial t}=\sum_{j=1}^{n}\Big\{-\left(\lambda_{0j}^{+}+\lambda_{0j}^{-}+\mu_j\right)\left[\varphi_i(z,t)-\varphi_i^{\{j\}}(z,t)\right]+\frac{\lambda_{0j}^{+}}{z_j}\left[\varphi_i(z,t)-\varphi_i^{\{j\}}(z,t)\right]+$$
$$+\left(\mu_j p_{j0}+\lambda_{0j}^{-}\right)z_j\varphi_i(z,t)\Big\}+\frac{\lambda_{0i}^{+}}{z_i}\left[\varphi_i(z,t)-\varphi_i^{\{i\}}(z,t)\right]+\left(\mu_i p_{i0}+\lambda_{0i}^{-}\right)z_i\varphi_i(z,t)+$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\Big\{\frac{\mu_j p_{ji}^{+}z_j}{z_i}\left[\varphi_i(z,t)-\varphi_i^{\{i\}}(z,t)\right]+\frac{\mu_i p_{ij}^{+}z_i}{z_j}\left[\varphi_i(z,t)-\varphi_i^{\{j\}}(z,t)\right]+ \qquad (12)$$

$$+\mu_i p_{ij}^{-}\left[\varphi_i(z,t)-\varphi_i^{\{i\}}(z,t)-\varphi_i^{\{j\}}(z,t)\right]\Big\}+\sum_{\substack{c,s=1,\\ c,s\neq i}}^{n}\left\{\frac{\mu_s p_{sc}^{+} z_s}{z_c}\left[\varphi_c(z,t)-\varphi_c^{\{c\}}(z,t)\right]+\right.$$

$$\left.+\mu_c p_{cs}^{-}\left[\varphi_i(z,t)-\varphi_i^{\{c\}}(z,t)-\varphi_i^{\{s\}}(z,t)\right]\right\}+\sum_{\substack{k_l=0\\ l=\overline{1,n}}}^{\infty}\alpha_i(k)\prod_{l=1}^{n} z_l^{k_l}.$$

Consider the last sum in (12). Suppose that incomes $r_i(k)$, $r_{ij}(k)$, $r_{0i}(k)$, $R_{i0}(k)$ do not depend on network states k, i.e. $r_i(k)=r_i$, $r_{ij}(k)=r_{ij}$, $r_{0i}(k)=r_{0i}$, $R_{i0}(k)=R_{i0}$. Since the $\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}\prod_{l=1}^{n} z_l^{k_l}=\prod_{l=1}^{n}\frac{1}{1-z_l}$,

in this case

$$\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty}\alpha_i(k)\prod_{l=1}^{n} z_l^{k_l}=\left\{r_i+\lambda_{0i}^{+}r_{0i}-\left(\mu_i p_{i0}+\lambda_{0i}^{-}\right)R_{i0}+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\mu_j p_{ji}^{+}r_{ji}-\mu_i p_{ij}^{+}r_{ij}-\mu_i p_{ij}^{-}r_{ij}\right]\right\}\prod_{l=1}^{n}\frac{1}{1-z_l}. \tag{13}$$

Let $i_1\neq i_2\neq\ldots\neq i_j$. We introduce some notation related to z-transforms of incomes of systems S_i:

$$\varphi_i^{\Omega}(z,t)=\varphi_i(z,t)\big|_{k_j=0,\,z_j=1,\,j=\overline{1,n}}=v_i(0,0,\ldots,0,t);$$

$$\varphi_i^{\Omega\setminus\{i_1\}}(z,t)=\varphi_i(z,t)\big|_{k_j=0,\,z_j=1,\,j=\overline{1,n},\ j\neq i_1}=\sum_{k_{i_1}=0}^{\infty}v_j\left(0,\ldots,0,k_{i_1},0,\ldots,0,t\right)z_{i_1}^{k_{i_1}},\ i_1=\overline{1,n};$$

$$\varphi_i^{\Omega\setminus\{i_1,i_2\}}(z,t)=\varphi_i(z,t)\big|_{k_j=0,\,z_j=1,\,j=\overline{1,n},\ j\neq i_1,i_2}=\sum_{k_{i_1}=0}^{\infty}\sum_{k_{i_2}=0}^{\infty}v_i\left(0,\ldots,0,k_{i_1},0,\ldots,0,k_{i_2},0,\ldots,0,t\right)z_{i_1}^{k_{i_1}}z_{i_2}^{k_{i_2}},$$

$$i_1,i_2=\overline{1,n},\ i_1\neq i_2;$$

. . .

$$\varphi_i^{\Omega\setminus\{i_1,i_2,\ldots,i_l\}}(z,t)=\varphi_i(z,t)\big|_{k_j=0,\,z_j=1,\,j=\overline{1,n},\,j\neq i_1,i_2,\ldots,i_l}=$$

$$=\sum_{k_{i_1}=0}^{\infty}\sum_{k_{i_2}=0}^{\infty}\ldots\sum_{k_{i_l}=0}^{\infty}v_i\left(0,\ldots,0,k_{i_1},0,\ldots,0,k_{i_2},0,\ldots,0,k_{i_l},0,\ldots,0,t\right)z_{i_1}^{k_{i_1}}z_{i_2}^{k_{i_2}}\cdot\ldots\cdot z_{i_j}^{k_{i_l}},$$

$$i_1,i_2,\ldots,i_l=\overline{1,n},\ i_1\neq i_2\neq\ldots\neq i_l;$$

. . .

$$\varphi_i^{\Omega\setminus\{i_1,i_2,\ldots,i_{n-1}\}}(z,t)=\varphi_i(z,t)\big|_{k_j=0,\,z_j=1,\,j=\overline{1,n},\,j\neq i_1,i_2,\ldots,i_{n-1}}=\varphi_i^{\{j\}}(z,t),\ i=\overline{1,n},$$

$$i_1\neq i_2\neq\ldots\neq i_{n-1}.$$

And in the function arguments v_i between k_{i_j} zeros can not stand, it is only important to $i_1 \neq i_2 \neq \ldots \neq i_j$. We present an algorithm for finding the expected incomes, using the relation (12).

3. Algorithm for finding of incomes

0-Step: it is necessary to define $\varphi_i^{\Omega}(z,t) = v_i(0,0,\ldots,0,t)$.

1st-Step: find z-transforms $\varphi_i^{\Omega \backslash \{i_1\}}(z,t)$, i.e. z-transformation of income of system S_i at moment time t, at the initial time it is located in a state $k = (0,0,...,k_{i_1},0,...,0)$. To do this, multiply the equation (3) on $\prod_{l=1}^{n} z_l^{k_l}$ and summing over all k_l, $l = \overline{1,n}$, from 0 to ∞. Taking into account that $k_l = 0,\ l \neq i_1$ the sums of (3)-(10) can be rewritten in the following form:

$$\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty} v_i(k,t) z_1^{k_1} z_2^{k_2} \cdot \ldots \cdot z_n^{k_n} = \begin{cases} 0,\ j \neq i_1, \\ \varphi_i^{\Omega \backslash \{i_1\}}(z,t),\ j = i_1, \end{cases},\ j = \overline{1,n}. \tag{14}$$

$$\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty} v_i(k - I_j, t) \prod_{l=1}^{n} z_l^{k_l} = \sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty} v_i(k,t) z_j \prod_{l=1}^{n} z_l^{k_l} = \begin{cases} 0,\ j \neq i_1, \\ z_{i_1} \varphi_j^{\Omega \backslash \{i_1\}}(z,t),\ j = i_1, \end{cases},\ j = \overline{1,n}. \tag{15}$$

$$\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty} v_i(k + I_j, t) \prod_{l=1}^{n} z_l^{k_l} = \begin{cases} 0, & j \neq i_1, \\ \dfrac{1}{z_{i_1}} \left(\varphi_i^{\Omega \backslash \{i_1\}}(z,t) - v_i(0,0,\ldots.,0,t) \right), & j = i_1, \end{cases},\ j = \overline{1,n}. \tag{16}$$

Obviously, it is equal to zero if $j \neq i_1$, because $k_j = 0$ at $j \neq i_1$. Consider this sum for $j = i_1$

$$\sum\nolimits_{4i_1}(z,t) = \sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty} v_i(k + I_i, t) \prod_{l=1}^{n} z_l^{k_l} z_{i_1} = z_{i_1} \sum\nolimits_{3i}(z,t) = 0. \tag{17}$$

If $i_1 = i$, then

$$\sum\nolimits_{5i}(z,t) = \sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty} v_i(k + I_j, t) \prod_{l=1}^{n} z_l^{k_l} z_i = z_i \sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty} v_i(k + I_j, t) \prod_{l=1}^{n} z_l^{k_l},\ j = \overline{1,n},\ j \neq i,$$

and taking into account (16),

$$\sum\nolimits_{5i}(z,t)=\frac{z_i}{z_j}\left[\varphi_i^{\Omega\setminus\{i\}}(z,t)-\left(\varphi_i^{\Omega\setminus\{i\}}(z,t)\right)^{\{j\}}\right].$$

And obviously, $\sum_{\substack{k_l=0,\\ l=\overline{1,n}}}^{\infty} v_i\left(k+I_j-I_i,t\right)\prod_{l=1}^{n} z_l^{k_l}=0$ for all $j=\overline{1,n},\ j\neq i$.

Similarly, we obtain that

$$\frac{z_s}{z_c}\left[\varphi_c^{\Omega\setminus\{c\}}(z,t)-\left(\varphi_c^{\Omega\setminus\{c\}}(z,t)\right)^{\{c\}}\right]=0,\ c,s=\overline{1,n},\ c,s\neq i. \tag{18}$$

$$\varphi_i^{\Omega\setminus\{i\}}(z,t)-\left(\varphi_i^{\Omega\setminus\{i\}}(z,t)\right)^{\{i\}}-\left(\varphi_i^{\Omega\setminus\{i\}}(z,t)\right)^{\{j\}}=0,\ j=\overline{1,n}. \tag{19}$$

$$\varphi_c^{\Omega\setminus\{c\}}(z,t)-\left(\varphi_c^{\Omega\setminus\{c\}}(z,t)\right)^{\{c\}}-\left(\varphi_c^{\Omega\setminus\{c\}}(z,t)\right)=0,\ c,s=\overline{1,n},\ c,s\neq i. \tag{20}$$

Consider the expression (13). Since the $j\neq i_1$ $\sum_{\substack{k_j=0,\\ j=\overline{1,n}}}^{\infty}\prod_{l=1}^{n} z_l^{k_l}=\sum_{k_{i_1}=0}^{\infty} z_{i_1}^{k_{i_1}}=\frac{1}{1-z_{i_1}}$,

in this case

$$R(z)=\sum_{\substack{k_j=0,\\ j=\overline{1,n}}}^{\infty}\alpha_i(k)\prod_{l=1}^{n} z_l^{k_l}=$$

$$=\begin{cases}\left(r_i+\lambda_{0i}^{+}r_{0i}-\left(\mu_i p_{i0}+\lambda_{0i}^{-}\right)R_{i0}+\mu_{i_1}p_{i_1 i}^{+}r_{i_1 i}-\mu_i\left(p_{ii_1}^{+}r_{ii_1}+p_{ii_1}^{-}r_{ii_1}\right)\right)\dfrac{1}{1-z_{i_1}},\ i_1\neq i,\\ =\left(r_i+\lambda_{0i}^{+}r_{0i}-\left(\mu_i p_{i0}+\lambda_{0i}^{-}\right)R_{i0}+\sum\limits_{\substack{j=1\\ j\neq i}}^{n}\left[\mu_j p_{ji}^{+}r_{ji}-\mu_i\left(p_{ij}^{+}r_{ij}+p_{ij}^{-}r_{ij}\right)\right]\right)\dfrac{1}{1-z_i},\ i_1=i.\end{cases} \tag{21}$$

Taking into account (14)-(21), the system of equations (1) after the above transformations we obtain for z-transforms $\varphi_i^{\Omega\setminus\{i_1\}}(z,t)$:

$$\begin{gathered}\frac{\partial\varphi_i^{\Omega\setminus\{i_1\}}(z,t)}{\partial t}=-\left(\lambda_{0i_1}^{+}+\lambda_{0i_1}^{-}+\mu_{i_1}\right)\varphi_i^{\Omega\setminus\{i_1\}}(z,t)+\left(\mu_{i_1}p_{i_1 0}+\lambda_{0i_1}^{-}\right)z_{i_1}\varphi_i^{\Omega\setminus\{i_1\}}(z,t)+\\ +\frac{\lambda_{0i_1}^{+}}{z_{i_1}}\left(\varphi_i^{\Omega\setminus\{i_1\}}(z,t)-v_i(0,0,....,0,t)\right)+R(z)\end{gathered} \tag{22}$$

where $R(z)$ defined in equation (21). Furthermore, by solving a system of differential equations (22) and expanding the functions $\varphi_i^{\Omega\setminus\{i_1\}}(z,t)$ in a power series by $z_{i_1}^{k_{i_1}}$, we can find the coefficients in these expansions $v_i(0,\ldots,0,k_{i_1},0,\ldots,0,t)$, $i_1=\overline{1,n}$.

2st-Step: find z-transforms $\varphi_i^{\Omega\setminus\{i_1,i_2\}}(z,t)$. For this recalculate all sums (3)-(10), (13), under the condition that $k_l=0,\ l\neq i_1,i_2$; expanding the found functions in a power series $z_{i_1}^{k_{i_1}}z_{i_2}^{k_{i_2}}$, find the coefficients in these expansions $v_i(0,\ldots,0,k_{i_1},0,\ldots,0,k_{i_2},0,\ldots,0,t)$;

– continuing in this way, on j-th step finding z-transforms $\varphi_i^{\Omega\setminus\{i_1,i_2,\ldots,i_j\}}(z,t)$, while $i_1,i_2,\ldots,i_j=\overline{1,n}$, $i_1\neq i_2\neq\ldots\neq i_j$; expansion coefficients $\varphi_i^{\Omega\setminus\{i_1,i_2,\ldots,i_j\}}(z,t)$ in a series in powers $z_{i_1}^{k_{i_1}}z_{i_2}^{k_{i_2}}\cdot\ldots\cdot z_{i_j}^{k_{i_j}}$ give us incomes $v_i(0,\ldots,0,k_{i_1},0,\ldots,0,k_{i_2},0,\ldots,0,k_{i_j},0,\ldots,0,t)$;

– further, continuing to make such steps, at $(n-1)$-th step find $\varphi_i^{\{j\}}(z,t)$, $j=\overline{1,n}$, and at n-th step - z-transforms $\varphi_i(z,t)$, which satisfies (11), expanding of through a series in powers $z_1^{k_1}z_2^{k_2}\cdot\ldots\cdot z_n^{k_n}$, we can find incomes $v_i(k_1,k_2,\ldots,k_n,t)$.

4. Example

Consider an open HM-network with negative messages, consisting of two $n=2$ QS: S_1 and S_2. The intensity of the input stream of positive and negative messages are equals, respectively: $\lambda_{01}^+=0{,}8$, $\lambda_{02}^+=3{,}2$, $\lambda_{01}^-=0{,}6$, $\lambda_{02}^-=1{,}4$. The intensity of service messages at network systems are equals: $\mu_1=2$, $\mu_2=4$. Let the transition probabilities of messages be equals, respectively: $p_{12}^+=0{,}2$, $p_{21}^+=0{,}6$, and probabilities, that the positive messages served in QS S_i, sent to QS S_j, as negative messages, equals: $p_{12}^-=0{,}1$, $p_{21}^-=0{,}3$. The probability of escape messages from the network to the external environment be equals: $p_{10}=0{,}7$, $p_{20}=0{,}1$. Let the incomes from network transitions between the states be equals: $r_1=r_2=5000$, $R_{10}=7000$, $R_{20}=4000$, $r_{01}=3000$, $r_{02}=2500$, $r_{12}=10000$, $r_{21}=12000$.

System (1) according to the entered parameters of the network will be:

$$\begin{aligned}\frac{dv_1(k,t)}{dt}&=-(4+2{,}6u(k_1)+4{,}4u(k_2))v_1(k,t)+\\&+1{,}6v_1(k_1+1,k_2,t)+3{,}2v_1(k_1,k_2+1,t)+\\&+(4u(k_1)+0{,}4(1-u(k_2)))v_1(k_1-1,k_2,t)+(0{,}9(1-u(k_1))+1.7u(k_2))v_1(k_1,k_2-1,t)+\\&+1{,}8u(k_2)v_1(k_1+1,k_2-1,t)+0{,}4u(k_1)v_1(k_1-1,k_2+1,t)+0{,}2v_1(k_1-1,k_2-1,t)+\\&+21600u(k_2)-16600u(k_1)-1000,\end{aligned}\tag{23}$$

$$\frac{dv_2(k,t)}{dt} = -(4 + 2{,}6u(k_1) + 4{,}4u(k_2))v_1(k,t) +$$
$$+ 0{,}8v_2(k_1+1,k_2,t) + 4v_2(k_1,k_2+1,t) +$$
$$+ (2u(k_1) + 0{,}2(1-u(k_2)))v_1(k_1-1,k_2,t) + (1{,}8(1-u(k_1)) + 2.4u(k_2))v_2(k_1,k_2-1,t) +$$
$$+ 0{,}4u(k_1)v_2(k_1+1,k_2-1,t) + 1{,}8u(k_2)v_1(k_1-1,k_2+1,t) + 0{,}9v_2(k_1-1,k_2-1,t) -$$
$$- 9800 + 8600u(k_1) - 21600u(k_2). \tag{24}$$

The ratios (12) can be rewritten as follows:

$$\frac{\partial\varphi_1(z,t)}{\partial t} = 11{,}2\varphi_1(z,t) - 3{,}6\varphi_1^{\{1\}}(z,t) - 7{,}8\varphi_1^{\{2\}}(z,t) + 2z_1\varphi_1^{\{1\}}(z,t) +$$
$$+ \frac{1{,}6(\varphi_1(z,t) - \varphi_1^{\{1\}}(z,t))}{z_1} + \frac{1{,}8z_2(\varphi_1(z,t) - \varphi_1^{\{1\}}(z,t))}{z_1} + \frac{3{,}2(\varphi_1(z,t) - \varphi_1^{\{2\}}(z,t))}{z_2} + \tag{25}$$
$$+ \frac{0{,}4z_1(\varphi_1(z,t) - \varphi_1^{\{2\}}(z,t))}{z_2} + \frac{15600}{(z_1-1)(z_2-1)} - 11600,$$

$$\frac{\partial\varphi_2(z,t)}{\partial t} = 11{,}9\varphi_2(z,t) - 2{,}5\varphi_2^{\{1\}}(z,t) - 6{,}7\varphi_2^{\{2\}}(z,t) + 1{,}7z_2\varphi_2^{\{1\}}(z,t) +$$
$$+ (2z_1 + 1{,}7z_2)\varphi_2(z,t) + \frac{0{,}8(\varphi_2(z,t) - \varphi_2^{\{1\}}(z,t))}{z_1} +$$
$$+ \frac{1{,}8z_2(\varphi_2(z,t) - \varphi_2^{\{1\}}(z,t))}{z_1} + \frac{3{,}2(\varphi_2(z,t) - \varphi_2^{\{2\}}(z,t))}{z_2} + \tag{26}$$
$$+ \frac{0{,}4z_1(\varphi_2(z,t) - \varphi_2^{\{2\}}(z,t))}{z_2} - \frac{28400}{(z_1-1)(z_2-1)} - 3900,$$

where:

$$\varphi_1^{\{1\}}(z,t) = \varphi_1^{\Omega\setminus\{2\}}(z,t) = v_1(0,k_2)z_2^{k_2}, \quad \varphi_1^{\{2\}}(z,t) = \varphi_1^{\Omega\setminus\{1\}} = v_1(k_1,0)z_1^{k_1},$$

$$\varphi_2^{\{1\}}(z,t) = \varphi_2^{\Omega\setminus\{2\}} = v_2(0,k_2)z_2^{k_2}, \quad \varphi_2^{\{2\}}(z,t) = \varphi_2^{\Omega\setminus\{1\}} = v_2(k_1,0)z_1^{k_1}.$$

Make use of the algorithm described above to find the expected income of system S_2.

Determine $\varphi_2(z,t) = v_2(0,0,t)$. Find z-transforms $\varphi_2^{\Omega\setminus\{i_1\}}$. For this we set $k_j = 0$, $j \neq i_1$ and multiply the system of equations (24) to $z_{i_1}^{k_{i1}}$ and summing over all k_{i_1}, $i_1 = 1,2$, from 0 to ∞. Using the sums (14)-(21), taking into account that $k_j = 0$, $j \neq i_1$, from (22) we obtain relation for z-transforms $\varphi_i^{\Omega\setminus\{i_1\}}(z,t)$:

$$\frac{\partial\varphi_2^{\Omega\backslash\{1\}}(z,t)}{\partial t}=-3{,}4\varphi_2^{\Omega\backslash\{1\}}(z,t)+2z_1\varphi_2^{\Omega\backslash\{1\}}(z,t)+$$
$$+0{,}8\frac{1}{z_1}\left(\varphi_2^{\Omega\backslash\{1\}}(z,t)-v_2(0,0,0,t)\right)+\frac{15600}{(z_1-1)(z_2-1)}-11600,$$

$$\frac{\partial\varphi_2^{\Omega\backslash\{2\}}(z,t)}{\partial t}=-8{,}6\varphi_2^{\Omega\backslash\{2\}}(z,t)+2{,}8z_2\varphi_2^{\Omega\backslash\{2\}}(z,t)+$$
$$+2{,}6\frac{1}{z_2}\left(\varphi_2^{\Omega\backslash\{2\}}(z,t)-v_2(0,0,0,t)\right)+\frac{28400}{(z_1-1)(z_2-1)}-3900.$$

The solution of this system of differential equations if we specify the initial conditions $\varphi_2^{\Omega\backslash\{1\}}(z,0)=\theta_1(z)$, $\varphi_2^{\Omega\backslash\{2\}}(z,0)=\theta_2(z)$, $v_2(0,0,0)=V$ was obtained in Mathematica and has the form:

$$\varphi_2^{\Omega\backslash\{1\}}(z,t)=e^{\frac{t(0,8+z_1(-3,4z_1+4,2)(z_2-1)-0,8z_2)}{z_1(1-z_1-z_2+z_1z_2)}}\left\{\frac{e^{\frac{t\left(0,8-4,2z_1+3,4z_1^2\right)}{z_1(z_1+1)}}}{(1-z_1-z_2+z_1z_2)\left(0{,}8-4{,}2z_1+3{,}4z_1^2\right)}\times\right.$$
$$\times\left[(z_1-1)(1-z_2)(0{,}8v_2(0,0,t)(z_1-1)+0{,}8V)\right.$$
$$\left.\left.+z_1\left(11600(z_1(1-z_2)+z_2)+0{,}8V+(11600-0{,}8V)z_2\right)\right]+\theta_1(z)\right\},$$

$$\varphi_2^{\Omega\backslash\{2\}}(z,t)=e^{\frac{t\left(2,8-11,4z_2+8,6z_2^2+z_1\left(-2,8+11,4z_2-8,6z_2^2\right)\right)}{z_2(1-z_1-z_2+z_1z_2)}}\left\{\frac{e^{\frac{t\left(2,8-11,4z_2+8,6z_2^2\right)}{z_2(z_2+1)}}}{(1-z_1-z_2+z_1z_2)\left(2{,}8-11{,}4z_2+8{,}6z_2^2\right)}\times\right.$$
$$\times\left[(z_2-1)(-2{,}8v_2(0,0,t)(z_1+1)(1-z_2)+z_2(3900(z_1(1-z_2)+z_2)-2{,}8V)+\right.$$
$$\left.\left.+2{,}8V-z_1(2{,}8V+(3900-2{,}8V)z_2))\right]+\theta_2(z)\right\}.$$

Initial conditions $\theta_1(z)$, $\theta_2(z)$ satisfy the following relations

$$\theta_1(z)=\varphi_2^{\Omega\backslash\{1\}}(z,0)=\sum_{k_1=0}^{\infty}v_2(k_1,0,0)z_1^{k_1}=\alpha z_1^{k_1^*},$$

where $v_2(k_1,0,0)=\begin{cases}\alpha,\ k_1=k_1^*,\\ 0,\text{in other cases,}\end{cases}$

$$\theta_2(z)=\varphi_2^{\Omega\backslash\{2\}}(z,0)=\sum_{k_2=0}^{\infty}v_2(0,k_2,0)z_1^{k_2}=\beta z_2^{k_2^*},$$

where $v_2(0,k_1,0)=\begin{cases}\beta,\quad k_2=k_2^*,\\ 0,\text{in other cases}\end{cases}.$

Substituting the expressions obtained for $\varphi_2^{\Omega_n \setminus \{i_1\}}(z,t)$, $i_1 = 1,2$ in (26), in the Mathematica package found an analytical solution of the differential equation $\varphi_2(z,t)$, with the initial condition $\varphi_2(z,0) = \overline{\theta}(z)$,

$$\overline{\theta}(z) = \varphi_2(z,0) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} v_2(k_1,k_2,0) z_1^{k_1} z_2^{k_2} = \gamma z_1^{k_1} z_2^{k_2},$$

where $v_2(k_1,k_2,0) = \begin{cases} \gamma, \; k_1 = k_1^*, \; k_2 = k_2^*, \\ 0, \text{in other cases} \end{cases}.$

The expected income $v_2(k_1, k_2, t)$ of system S_2 are coefficients of expansion of the function $\varphi_2(z,t)$ into a double series of powers $z_1^{k_1}$, $z_2^{k_2}$ and it was conducted in the Mathematica package of mathematical calculations.

Let $V = 0$, $k_1^* = k_2^* = 0$, $\alpha = \beta = \gamma = 1$, if the initial conditions are: $\theta_1(z) = 1$, $\theta_2(z) = 1$, $\overline{\theta}(z) = 1$. In Figure 1 there is a graph of income S_2 at $k_1 = 2$, $k_2 = 2$.

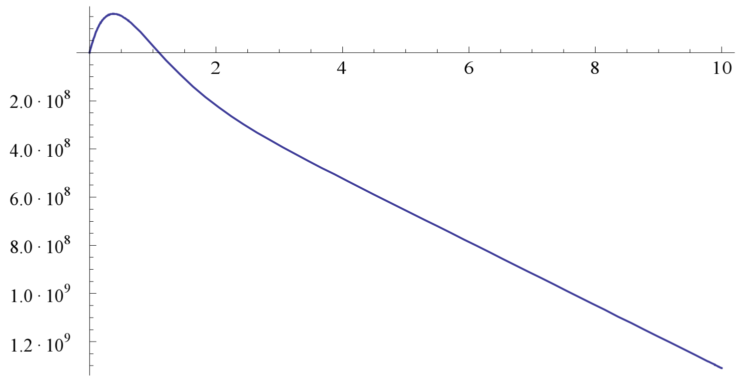

Fig. 1. Income chart $v_2(2,2,t)$ on interval [0,10]

Conclusions

The practical importance of the results lies in the fact that with the help of the proposed method one can find the expected incomes of systems considered network with an infinite number of states.

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