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STATIC ANALYSIS OF CIRCULAR AND ELLIPTIC PLATES RESTING ON INTERNAL FLEXIBLE SUPPORTS BY THE BOUNDARY ELEMENT METHOD

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Abstract. A static analysis of circular and elliptic Kirchhoff plates resting on internal elastic supports by the Boundary Element Method is presented in the paper. Elastic support has the character of Winkler-type elastic foundations. Bilateral and unilateral internal constraints are taken into consideration. The Betti's theorem is used to derive the boundary-domain integral equation. The direct version of the boundary element method is presented and simplified boundary conditions, including curvilinear boundary elements, are introduced. The collocation version of boundary element method with non-singular approach is presented.

Keywords: *internally supported plates, Boundary Element Method, fundamental solutions, curvilinear elements*

Introduction

Plates resting on internal flexible constraints are often used in building structures. The analysis of internally supported plates in terms of the Boundary Element Method (BEM) has been the subject of numerous studies, e.g. [1-4]. The governing equation was formulated and derived using the direct approach. The BEM is the alternative way to the most popular Finite Element Method [5]. The BEM is often used in the theory of plates and is particularly suitable to analyse the plates of arbitrary shapes. The main advantage of BEM is its relative simplicity of formulating and solving problems of the potential theory and the theory of elasticity. Burczyński [6] described the BEM in a comprehensive manner and its application in a variety of fields, the theory of elasticity together with the appropriate solutions and a discussion of the basic types of boundary elements. Similarly, Wrobel and Aliabadi [7] presented applications of BEM in a wide range. Consideration of internal constraints requires modification of the boundary integral equations. According to the Bèzine approach [2], additional internal collocation points are introduced in which the forces or displacements are treated as unknown variables. This entails transformation of the pure boundary integral equation to boundary-domain integral equation. An alternative coupled BEM-flexibility force method was proposed by Rashed [4]. The major drawback of this approach is the necessity of the support to the edge of the plate providing geometric invariability of structure. In order to simplify the calculation procedures Guminiak *et al*. [8] proposed an alternative formulation of the boundary-domain integral equation for a thin plate. The authors used the Bèzine technique to establish deflections and forces of support reaction in internal collocation points. Katsikadelis *et al*. [9] used direct BEM approach to solve static and dynamic problem of plates with support condition inside a domain. Pawlak and Guminiak [10] applied the BEM and the FSM to solve similar problems considering unilateral internal constraints. Katsikadelis [11] described an application of BEM in a wide aspects of engineering analysis of the plates. The author also applied the Analog Equation Method (AEM) formulation in terms of BEM. The AEM approach also was used by Guminiak and Litewka [12] for rectangular thin plates resting on Winkler-type elastic foundation.

The present paper is devoted to application of BEM considering simplified boundary conditions for bending analysis of thin circular and elliptic plates resting on internal flexible support. In this approach there is no need to introduce the equivalent shear forces at the boundary and concentrated forces at the plate corners. Internal elastic support was introduced using the Bèzine technique.

1. Integral formulation of thin plate bending

A static problem of a plate resting on an internal flexible support is considered. Internal support has a discrete character. On the plate boundary, the following variables are considered: shear force $\tilde{T}_n$, bending moment M_n and deflection w, angle of rotation in normal direction φ_n and angle of rotation in tangent direction φ_s. The expression $\tilde{T}_n(\mathbf{y}) = T_n(\mathbf{y}) + R_n(\mathbf{y})$ denotes shear force for clamped and simply-supported edges

$$\tilde{T}_n(\mathbf{y}) = \begin{cases} V_n(\mathbf{y}) & \text{on the boundary far from the corner} \\ R_n(\mathbf{y}) & \text{on a small fragment of the boundary close to the corner} \end{cases} \tag{1}$$

Because the relation between $\varphi_s(\mathbf{y})$ and the deflection is known: $\varphi_s(\mathbf{y}) = dw(\mathbf{y})/ds$ can be evaluated using a finite difference scheme of the deflection with two or more adjacent nodal values. In this analysis, the employed finite difference scheme includes the deflections of two adjacent nodes. The boundary-domain integral equations are derived using the Betti's theorem. Two plates are considered: the infinite plate, subjected unit concentrated loading and the real one. As a result, the first boundary-domain integral equation is in the form:

$$
\begin{aligned}
&c(\mathbf{x})\cdot w(\mathbf{x})+\int_{\Gamma}\left[T_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-M_{ns}^*(\mathbf{y},\mathbf{x})\cdot\frac{dw(\mathbf{y})}{ds}-M_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})\right]\cdot\; d\Gamma(\mathbf{y})=\\
&=\int_{\Gamma}\left[\widetilde{T}_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\varphi_n^*(\mathbf{y},\mathbf{x})\right]\cdot d\Gamma(\mathbf{y})+\int_{\Omega}p(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})\cdot d\Omega(\mathbf{y})-\sum_{n=1}^{N}S_n\cdot w^*(n,\mathbf{x})
\end{aligned}
\tag{2}
$$

where the fundamental solution of biharmonic equation $\nabla^4 w=(1/D)\overline{\delta}(\mathbf{y}-\mathbf{x})$ is given as a Green function

$$
w^*(\mathbf{y},\mathbf{x})=\frac{1}{8\pi D}r^2\ln r
\tag{3}
$$

for a thin isotropic plate, S_n expresses internal support reaction specified in internal collocation point, $r=|\mathbf{y}-\mathbf{x}|$, $\overline{\delta}$ is Dirac delta and $D=\left(E\,h_{\mathrm{p}}^3\right)/\left(12\,\left(1-\nu_{\mathrm{p}}^2\right)\right)$ is a plate stiffness. The coefficient $c(\mathbf{x})$ depends on the localization of point $\mathbf{x}$ and $c(\mathbf{x})=1$, when $\mathbf{x}$ is located inside the plate region, $c(\mathbf{x})=0.5$, when $\mathbf{x}$ is located on the smooth boundary and $c(\mathbf{x})=0$, when $\mathbf{x}$ is located outside the plate region.

The second boundary integral equation can be derived by substituting a unit concentrated force $P^*=1$ by unit concentrated moment $M_n^*=1$. It is equivalent to differentiate the first boundary-domain integral equation (2) on n direction in point $\mathbf{x}$ on a plate boundary.

$$
\begin{aligned}
&c(\mathbf{x})\cdot\varphi_n(\mathbf{x})+\int_{\Gamma}\left[\overline{T}_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-\overline{M}_{ns}^*(\mathbf{y},\mathbf{x})\cdot\frac{dw(\mathbf{y})}{ds}-\overline{M}_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})\right]\cdot\; d\Gamma(\mathbf{y})=\\
&=\int_{\Gamma}\left[\widetilde{T}_n(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\right]\cdot d\Gamma(\mathbf{y})+\int_{\Omega}p(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x})\cdot d\Omega(\mathbf{y})-\sum_{n=1}^{N}S_n\cdot\overline{w}^*(n,\mathbf{x})
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
&\left\{\overline{T}_n^*(\mathbf{y},\mathbf{x}),\overline{M}_n^*(\mathbf{y},\mathbf{x}),\overline{M}_{ns}^*(\mathbf{y},\mathbf{x}),\overline{w}^*(\mathbf{y},\mathbf{x}),\overline{\varphi}_n^*(\mathbf{y},\mathbf{x}),\overline{\varphi}_s^*(\mathbf{y},\mathbf{x})\right\}\;=\\
&=\frac{\partial}{\partial n(\mathbf{x})}\left\{T_n^*(\mathbf{y},\mathbf{x}),M_n^*(\mathbf{y},\mathbf{x}),M_{ns}^*(\mathbf{y},\mathbf{x}),w^*(\mathbf{y},\mathbf{x}),\varphi_n^*(\mathbf{y},\mathbf{x}),\varphi_s^*(\mathbf{y},\mathbf{x})\right\}
\end{aligned}
$$

2. Types of boundary elements

In the simplest approach, the boundary element of the constant type is introduced (Fig. 1a). It is also possible to define the geometry of the element considering three nodal points and only one collocation point connected with the relevant physical boundary value (Fig. 1b). The collocation point may be located slightly outside of a plate edge. The geometry of the element can be defined using polynominal function, described in standard coordinate system $\langle -1,\ 0,\ 1\rangle$. These functions are in the form:

$$N_1 = \frac{1}{2}\eta \cdot (\eta - 1),\ N_2 = 1 - \eta^2,\ N_3 = -\frac{1}{2}\eta \cdot (\eta - 1) \tag{5}$$

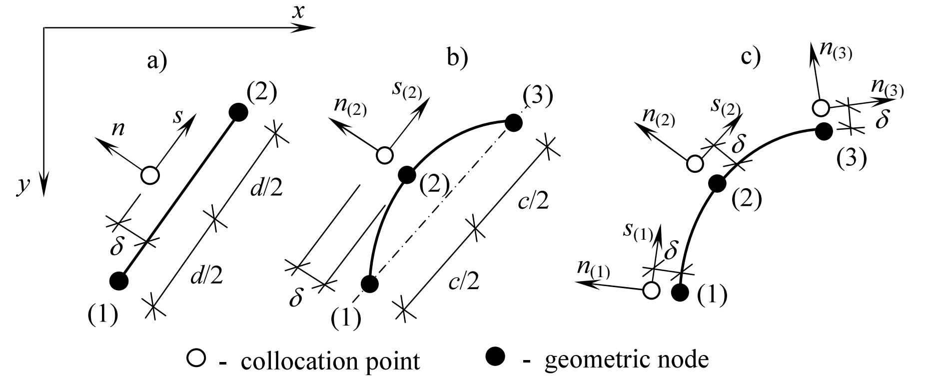

Fig. 1. Boundary elements of the constant type in non-singular approach

A quadratic curvilinear isoparametric element is shown in Figure 1c. According to the non-singular approach, the boundary (boundary-domain) integral equations can be formulated using the approach of single collocation point associated with each boundary element of the constant type and single collocation point associated with each geometric node of the quadratic element.

3. Assembly of the set of algebraic equation

Let it be assumed that a plate boundary is discretized using constant elements. Internal flexible support can be treated as the Winkler-type foundations, where the support reaction S_n can be expressed in the simple form

$$S_n = k_n \cdot w_n \tag{6}$$

where k_n and w_n are the support stiffness and displacement. In the case of the free edge, the characteristic matrix must be expanded using additional components $\mathbf{G_{BS}}$ and $\mathbf{\Delta}$:

$$\begin{bmatrix} \mathbf{G_{BB}} & \mathbf{G_{BS}} & \mathbf{G_{Bw}} \\ \mathbf{\Delta} & -\mathbf{I} & \mathbf{0} \\ \mathbf{G_{wB}} & \mathbf{G_{wS}} & \mathbf{G_{ww}} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{B} \\ \varphi_{\mathbf{s}} \\ \mathbf{w} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F_B} \\ \mathbf{0} \\ \mathbf{F_w} \end{Bmatrix} \tag{7}$$

and **B** is the vector of boundary independent variables, φ_s is the vector of additional parameters of the angle of rotation in the tangential direction, which depend on the boundary deflection in case of the free edge, $\mathbf{G_{BB}}$ is the matrix grouping boundary integrals dependent on type of boundary. Matrix $\mathbf{G_{BS}}$ groups boundary integrals of functions M_{ns}^* and $\overline{M}_{ns}^*$ in case of free edge occurence and it is the additional matrix grouping boundary integrals corresponding with rotation in tangential direction φ_s. The matrix $\mathbf{G_{Bw}}$ groups values of fundamental functions w^* and $\overline{w}^*$ established in internal collocation points associated with internal constraints. The matrix $\mathbf{\Delta}$ groups the finite difference expressions for the angle of rotation in the tangential direction φ_s in terms of deflections at suitable, adjacent nodes and **I** is the unit matrix. In the computer program deflections at two neighbouring nodes are used. Hence, for a clamped edge, a simply-supported edge and a free edge, two independent unknowns are always considered. Matrices $\mathbf{G_{wB}}$, $\mathbf{G_{wS}}$ and $\mathbf{G_{ww}}$ group boundary integrals and values of fundamental function w^* calculates in collocation points associated with internal supports respectively. All of the designations are shown in Figure 2.

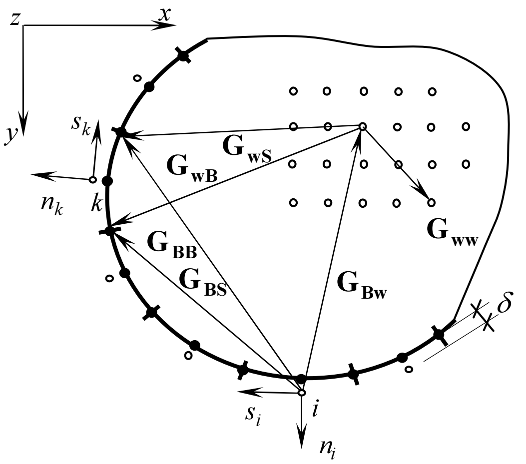

Fig. 2. Construction of characteristic matrix

3.1. Construction of characteristic matrix

The boundary-domain integral equation will be formulated in a non-singular approach. To construct the characteristic matrix **G**, integration of suitable fundamental function on boundary is needed. Integration is done in a local coordinate system n_i, s_i connected with i^{th} boundary element and next, these integrals must be transformed to n_k, s_k coordinate system, connected with k^{th} element. Localization of collocation point is defined by the parameter δ or non-dimensional parameter ε.

This parameter can be defined as $\varepsilon = \delta/d$ or $\varepsilon = \delta/c$ (Fig. 1). To calculate elements of the characteristic matrix there are applied the following methods: *a*) classic, numerical Gauss procedure for non-quasi diagonal elements or *b*) modified, numerical integration of Gauss method for quasi-diagonal elements proposed by Litewka and Sygulski [13]. The authors proposed inverse localization of the Gauss points in domain of integration, which is illustrated in Figure 3. Boundary integrals on curved element are calculated according to Gauss method. Integrals of fundamental functions over the plate edge are calculated using n_i, s_i coordinate system, connected with i^{th} physical node. Then, they are transformed to n_k, s_k coordinate system [8, 14, 15].

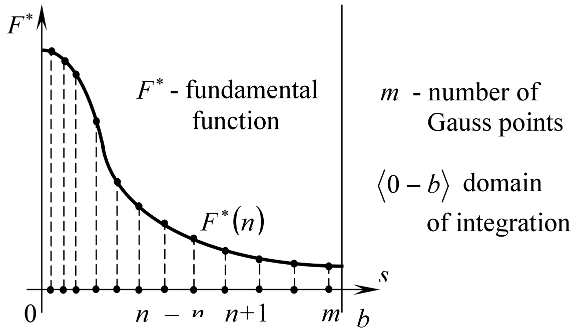

Fig. 3. Calculation of quasi-diagonal integrals using modified Gauss method [13]

In case of consideration of a free edge, the angle of rotation in a tangent direction can be expressed by deflection of two neighbouring nodes

$$\varphi_s^{(i-1)} = \varphi_s^{(i)} = \varphi_s^{(i+1)} = \left(w_b^{(i+1)} - w_b^{(i)}\right)/d_{i+1} \tag{8}$$

where d_i is the projection of a section connecting physical nodes (collocation points) i and $i + 1$ on the line tangential to the boundary element in collocation point i^{th}. It is also assumed that a plate can be supported on boundary and also resting on unilateral internal supports. In the last case, especially when a plate is supported only inside its domain, the problem can be solved iteratively. On each iteration step the collocation points inside a plate with a negative value of support reaction are switched off. The process can be stopped, when between the iterations there is no change of sign of any reaction.

3.2. Construction of right-hand-side vector

It is assumed that constant loading p is acting on a plate surface. Integrals $p\int_\Omega w^* d\Omega$ and $p\int_\Omega \overline{w}^* d\Omega$ can be evaluated analytically in terms of the Abdel-Akher and Hartley proposition (contour of loading is expressed in polygonal form) [16].

4. Calculation of deflection, angle of rotation, bending and torsional moments inside a plate domain

The solution of algebraic equations allows one to determine the boundary variables. Then, it is possible to calculate the deflection, angle of rotation in an arbitrary direction, bending and torsional moments at an arbitrary point of the plate domain. Each value can be expressed as the sum of three variables depending on the boundary variables $\overline{\mathbf{B}}$, external loading p and reaction of internal supports **S**, for example deflection

$$w = w\left(\overline{\mathbf{B}}\right) + w\left(p\right) + w\left(\mathbf{S}\right) \tag{9}$$

where $\overline{\mathbf{B}} = \{\mathbf{B}\ \varphi_{\mathrm{S}}\}^T$. A similar relation can be applied to establish the angle of rotation in an arbitrary direction. In terms of the thin plate theory, the bending moments and torsional moment are given in the classic form $M_x(x,y) = -D(w,_{xx} + \nu w,_{yy})$, $M_y(x,\mathrm{y}) = -D(w,_{yy} + \nu w,_{xx})$ and $M_{xy}(x,y) = -D(1-\nu)\cdot w,_{xy}$ and $w(x,y)$ is the function of displacements and x, y are the global coordinates of an arbitrary point.

5. Numerical examples

Circular and elliptic plates with various boundary conditions are considered. Plates are subjected only to a uniformly distributed loading p = 1.0 kN/m on the entire surface or concentrated force P = 10.0 kN at its centre. Twenty Gauss points are applied to evaluate boundary integrals. Circular plates are divided by boundary elements with the same length. For elliptic plate localization of geometrical nodes on the edge for 32 boundary elements is presented in Figure 4. For 64 boundary elements, similar localization is assumed, dividing all of segments: *l*, *l*/2, *l*/3 and *l*/6 by halves.

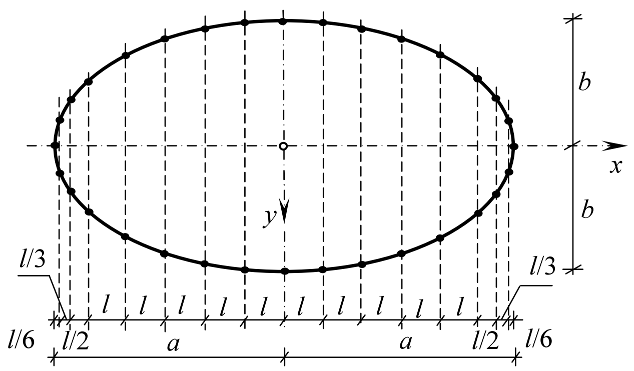

Fig. 4. Localization of boundary elements inscribed in ellipse contour

Localization of internal collocation points corresponding to the number of 128 discrete supports localized inside a plate domain which for circular plates is shown in the Figure 5.

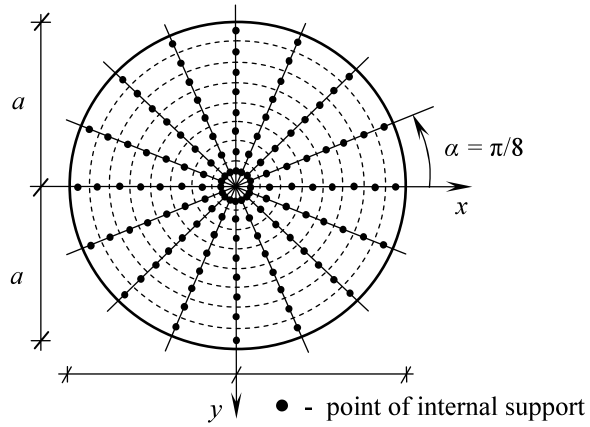

Fig. 5. Localization of internal collocation points for a circular plate

For an elliptic plate, coordinates of internal collocation point are assumed according to polar coordinates. The radius to the selected collocation point is expressed as follows:

$$r = \sqrt{b^2 / \left(1 - e^2 \cdot \cos^2 \alpha\right)} \tag{10}$$

where $e^2 = \left(a^2 - b^2\right) / a^2$. In the considered examples, angle α is the multiple of $\pi/8$ rad. Localization of internal collocation points for number of 128 discrete supports is presented in Figure 6.

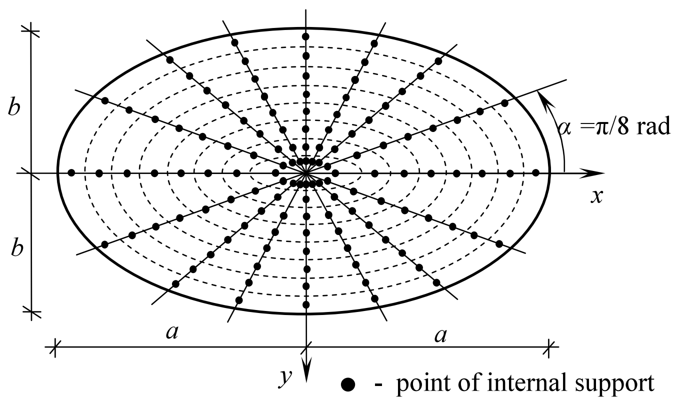

Fig. 6. Localization of internal collocation points for an elliptic plate

The following plate properties are assumed: E = 205.0 GPa and v = 0.3, thickness h = 0.01 m. Internal support stiffness k_i = 50.0 kN/m. The circular plate radius is equal to a = 2.0 m. Elliptic plate half-axes are equal to a = 3.0 m and b = 2.0 m. A numerical analysis was conducted using following boundary and finite element discretization: BEM I - rectilinear boundary element of the constant type, $\varepsilon = \delta/d = 0.01$; BEM II - curved, simplified boundary element of the constant type, $\varepsilon = \delta/c = 0.1$; BEM III - three-node isoparametric curved boundary element, $\varepsilon = \delta/c = 0.1$; FEM - finite element analysis was carried out using Abaqus/STANDARD v6.12 computational program and eight-node doubly curved shell finite element with reduced integration (S8R) was adopted [17]. The plate domain was divided into 3936 and 4706 elements for a circular and elliptic plate, respectively. The results of calculation for circular plate clamped on a whole edge, resting on internal supports and subjected to the uniformly distributed loading p are presented in Tables 1 and 2.

Table 1

Deflection at the plate centre

Number of boundary elements	$\widetilde{w} = wD/pa^4$		
	BEM I	BEM III	FEM
32	$9.8428 \cdot 10^{-4}$	$9.8438 \cdot 10^{-4}$	$9.8558 \cdot 10^{-4}$
64	$9.8476 \cdot 10^{-4}$	$9.8480 \cdot 10^{-4}$	

Table 2

Bending moments

Number of boundary elements	$\widetilde{M} = M_r/pa^2$			
	At the centre		On boundary	
	BEM I	BEM III	BEM I	BEM III
32	–0.021265	–0.021290	–0.040548	–0.040468
64	–0.021424	–0.021430	–0.040333	–0.040317

Results of calculation for circular plate simply-supported on a whole edge, resting on internal supports and subjected to the uniformly distributed loading p are presented in Tables 3 and 4.

Table 3

Deflection at the plate centre

Number of boundary elements	$\widetilde{w} = wD/pa^4$		
	BEM I	BEM II	FEM
32	$1.0148 \cdot 10^{-3}$	$1.0150 \cdot 10^{-3}$	$1.0137 \cdot 10^{-3}$
64	$1.0136 \cdot 10^{-3}$	$1.0137 \cdot 10^{-3}$	

Table 4

Bending moment at the plate centre

Number of boundary elements	$\widetilde{M} = M_r / pa^2$	
	BEM I	BEM II
32	–0.029675	–0.029704
64	–0.029809	–0.029822

Results of calculation for circular plate with a whole edge free, resting on internal supports and subjected to the concentrated force P at the centre are presented in Table 5 and Figure 7.

Table 5

Deflection at the plate centre

Number of boundary elements	$\widetilde{w} = wD/Pa^2$			
	Bilateral		Unilateral	Bilateral
	BEM I	BEM II	BEM I	FEM
32	$2.8556 \cdot 10^{-3}$	$2.8557 \cdot 10^{-3}$	$2.9135 \cdot 10^{-3}$	$2.8676 \cdot 10^{-3}$
64	$2.8558 \cdot 10^{-3}$	$2.8558 \cdot 10^{-3}$	$2.9146 \cdot 10^{-3}$	

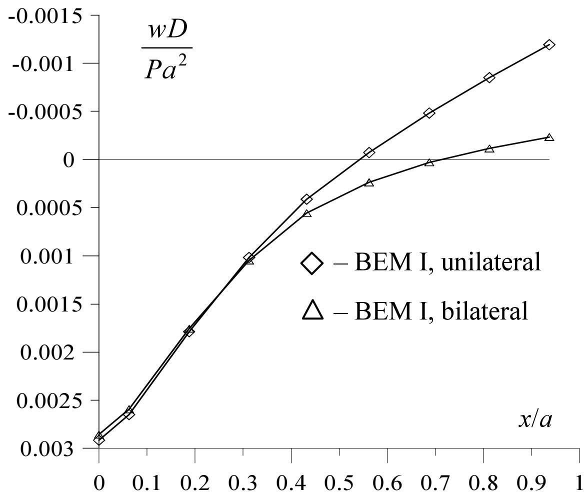

Fig. 7. Deflection along plate radius r/a

Results of calculation for an elliptic plate with a whole edge free, resting on internal supports and subjected to the concentrated force P at the centre are presented in Table 6 and Figure 8.

Table 6

Deflection at the plate centre

Number of boundary elements	$\widetilde{w} = wD/Pb^2$			
	Bilateral		Unilateral	Bilateral
	BEM I	BEM II	BEM I	FEM
32	$3.2067\cdot10^{-3}$	$3.1256\cdot10^{-3}$	$3.2773\cdot10^{-3}$	$3.1116\cdot10^{-3}$
64	$3.2067\cdot10^{-3}$	$3.2266\cdot10^{-3}$	$3.2773\cdot10^{-3}$	

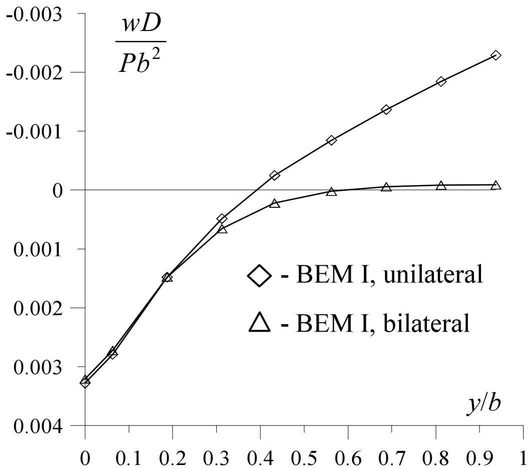

Fig. 8. Deflection along half-axis *x/b*

Conclusions

Static analysis of plates resting on internal elastic supports using the Boundary Element Method is presented. The problem was solved using the Kirchhoff theory of plates with the modified approach, in which the boundary conditions are defined so that there is no need to introduce equivalent boundary quantities dictated by the boundary value problem for the biharmonic differential equation even if typical rectilinear boundary element was used. The Bèzine technique and static fundamental solution for a usual thin plate were used to introduce supports inside a plate domain and establish their reactions and deflections. The modified Gauss method [13] was used to calculate quasi-diagonal integrals for curved boundary elements. The application of curved boundary elements gives a similar result as in the case of typical rectilinear elements. Direct collocation non-singular BEM formulation of plate bending considering internal supports in case of a large number of unknowns may lead to wrong conditioning of the characteristic matrix. The analysis of influence of collocation point localization on conditioning of characteristic matrix was carried out in [18].

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