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THE SYSTEM M_2^0/G/1/m WITH THRESHOLD CONTROL OF THE ARRIVAL RATE AND SERVICE TIME

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Abstract. We consider a $M_2^0/G/1/m$ queueing system with arrival of customer batches, which uses a threshold control mechanism of the service time and arrival rate. The system receives two independent flows of customers, one of which is blocked in an overload mode (under the condition that the number of customers in the system exceeds a given threshold value h). Full blocking of the input flow is carried out from the moment when the queue length reaches the number m until the beginning of the service of the first customer, for which the number of customers in the system does not exceed h. From the beginning of the service of the first customer during the excess of number of customers in the system of h until the completion of full blocking the time of service of customer is distributed under the law of $\widetilde{F}(x)$ (an increased service rate is used). Rest of the time the system applies the normal service rate with the distribution function $F(x)$ of service time. Laplace transforms for the distributions of the number of customers in the system during the busy period and for the distribution function of the busy period are found. The average duration of the busy period is obtained. Formulas for the stationary distribution of the number of customers in the system, for the probability of service and for the stationary characteristics of the system are established. The obtained results are verified with the help of a simulation model constructed with the assistance of GPSS World tools.

Keywords: *queueing system, flows of two types of customers, batch arrival of customers, threshold control, busy period, distribution of the number of customers*

Introduction

For the purpose of preventing overloads in the information and telecommunication systems a control both of an input flow and its parameters, and service rate is used. According to [1-3], the queuing systems with threshold control may be adequate models for evaluating the quality of functioning of SIP servers under overloads.

A large number of publications, in particular articles [4-8], are devoted to the study of queueing systems with threshold strategies of functioning. Most studies examined a single-channel system with an arbitrary distribution of the service time.

In this paper we consider a $M_2^\theta/G/1/m$ queuing system with the independent flows of two types of customers. In this system the control is applied both to the parameters of these flows and to the service rate.

Constructions of queueing systems which are closest to the considered system are discussed in [4, 5, 8]. System with flows of two types of customers is also studied in [4, 8], and in [5] a similar mechanism of full blocking of the flow of customers is used.

In contrast to [4], in the studied system a batch arrival of customers is provided, the switching of the service rate is applied and another threshold control mechanism is used. In contrast to this article, the mode of partial blocking of the input flow is not considered in paper [5], and in [8] a hysteretic control mechanism of the input flow intensity for a multi-channel system $M_2^X/M/n$ is applied.

1. Description of the system

Let us consider a $M_2^\theta/G/1/m$ queueing system that receives independent flows of two types of customer batches and that is formally described as follows. Let sequences of random variables $\{\alpha_{1n}\}$, $\{\alpha_{2n}\}$, $\{\theta_n\}$, $\{\beta_n\}$, $\{\tilde{\beta}_n\}$ $(n \geq 1)$ be specified, where α_{in} is the time between arrivals of the $(n-1)$-th and n-th batches of the flow number i $(i=1,2)$, θ_n is the number of customers in the n-th batch, and β_n and $\tilde{\beta}_n$ are the service time of the n-th customer in the normal service mode and in the mode of full blocking respectively. All these random variables are supposed to be totally independent and $\mathbf{P}\{\alpha_{in} < x\} = 1 - e^{-\lambda_i x}$ $(\lambda_i > 0;\ i = 1,2)$; $\mathbf{P}\{\theta_n = k\} = a_k$ $(k \geq 1)$, $\sum_{k=1}^{\infty} a_k = 1$; $\mathbf{P}\{\beta_n < x\} = F(x)$ $(x \geq 0)$, $F(0) = 0$ and $\mathbf{P}\{\tilde{\beta}_n < x\} = \tilde{F}(x)$ $(x \geq 0)$, $\tilde{F}(0) = 0$. If $\mathbf{P}\{\theta_n = 1\} = a_1 = 1$, then customers arrive in the system one by one (this is the ordinary flow).

Thus, the time intervals between the moments of arrival of customers batches of the flow number i are independent random variables distributed exponentially with parameter λ_i ($i = 1,2$). In the total flow being a superposition of the first and second type of flows, the time intervals between the moments of arrival of customers batches have the exponential distribution with parameter $\lambda = \lambda_1 + \lambda_2$ [9, p. 83].

Customers are served one by one, a served customer leaves the system, and the server immediately starts serving a customer from the queue, if one exists, or waits for the arrival of the next customer batch. The first-in first-out (FIFO) service discipline is used. A queue inside one customer batch can be arbitrarily organized, since the characteristics under study are independent of the way in which the queue is organized.

Let m be the maximum number of customers that can simultaneously be in the queue. Thus, if a batch of θ_n customers arrives in the system containing $k \in [0, m+1]$ customers, then, only $\min\{\theta_n, m+1-k\}$ of these customers join the queue, the remaining ones being lost.

There are three modes of control of the input flow intensity: normal mode, partial blocking mode and full blocking mode. In the normal mode customers of both types are accepted for service and $F(x)$ is the distribution function of the service time of each customer. In the partial blocking mode the acceptance of customers of the second flow stops and customers of the first flow are accepted only. In the full blocking mode accept of all customers stops. In the partial and full blocking modes the service time is distributed according to the law of $\widetilde{F}(x)$.

Let us describe the mechanism of switching modes. Let h be a given number ($2 \le h \le m-2$). Denote the number of customers in the system at time t as $\xi(t)$ and let t_b be a moment of beginning of a customer service. If $\xi(t_b) \le h$, then during the service of this customer the normal mode is applied. Switching to the partial blocking mode takes place at the moment t_b of service beginning the first customer for which the inequalities $h+1 \le \xi(t_b) \le m$ hold. The full blocking mode is activated in the moment of reaching of the queue length of m and continues until the moment t_b of the service beginning the customer for which the equality $\xi(t_b) = h$ holds. Switching to the service mode with the function of the service time distribution of $\widetilde{F}(x)$ is carried out not at the start of the the partial blocking mode, but at the beginning of the first customer service during the term of this mode.

The assumptions $2 \le h \le m-2$ are introduced only in order not to consider cases for which the formulas are different from those shown here, and in no way detract from the generality of the obtained results.

To study the probability characteristics of the described queueing system we use an approach based on the potential method proposed by V. Korolyuk [10]. This approach was previously used by us, in particular in the works [5, 6, 11].

2. Basic notations and auxiliary results

Denote by $\mathbf{P}_n$ the conditional probability, provided that at the initial time the number of customer in the queueing system is $n \ge 0$, and by $\mathbf{E}$ ($\mathbf{P}$) the conditional expectation (the conditional probability) if the system starts to work at the time of arrival of the first batch of customers. We introduce the following notations: $\eta(x, \lambda)$ is the number of customers arriving in the system during the time interval

$[0;x)$ under the condition that the time intervals between the moments of arrival of the batch of customers is exponentially distributed with parameter λ; a_i^{k*} is the k-fold convolution of the sequence a_i; $a(s,z)=s+\lambda(1-\alpha(z))$; $a_1(s,z)=s+$ $+\lambda_1(1-\alpha(z))$. Let

$$f(s)=\int_0^\infty e^{-sx}dF(x),\qquad \widetilde{f}(s)=\int_0^\infty e^{-sx}d\widetilde{F}(x);$$

$$M=\int_0^\infty x\,dF(x)<\infty;\qquad \widetilde{M}=\int_0^\infty x\,d\widetilde{F}(x)<\infty;\qquad e_a=\sum_{k=1}^\infty ka_k<\infty;$$

$$\overline{F}(x)=1-F(x),\qquad \overline{\widetilde{F}}(x)=1-\widetilde{F}(x);\qquad \alpha(z)=\sum_{k=0}^\infty z^k a_k;$$

$$\overline{a}_n=\sum_{k=n}^\infty a_k,\qquad \overline{p}_n(s)=\sum_{k=n}^\infty p_k(s),\qquad \overline{q}_n(s)=\sum_{k=n}^\infty q_k(s).$$

We specify the sequences $p_i(s)$, $\tilde{p}_i(s)$ $(\mathrm{Re}\, s\ge 0)$ as

$$p_i(s)=\frac{1}{f(s)}\int_0^\infty e^{-sx}\mathbf{P}\{\eta(x,\lambda)=i+1\}dF(x)=\frac{1}{f(s)}\sum_{k=0}^{i+1}a_{i+1}^{k*}\int_0^\infty e^{-(\lambda+s)x}\frac{(\lambda x)^k}{k!}dF(x);$$

$$\tilde{p}_i(s)=\frac{1}{\widetilde{f}(s)}\int_0^\infty e^{-sx}\mathbf{P}\{\eta(x,\lambda_1)=i+1\}d\widetilde{F}(x)= \tag{1}$$

$$=\frac{1}{\widetilde{f}(s)}\sum_{k=0}^{i+1}a_{i+1}^{k*}\int_0^\infty e^{-(\lambda_1+s)x}\frac{(\lambda_1 x)^k}{k!}d\widetilde{F}(x),\qquad i\ge -1.$$

The sequences of functions $R_k(s)$ and $\widetilde{R}_k(s)$ $(k\ge 1)$ are defined by the equalities

$$\sum_{k=1}^\infty z^k R_k(s)=\frac{z}{f(a(s,z))-z},\ |z|<\nu(s);\quad \sum_{k=1}^\infty z^k \widetilde{R}_k(s)=\frac{z}{\widetilde{f}(a_1(s,z))-z},\ |z|<\tilde{\nu}(s), \tag{2}$$

where $\nu(s)$ and $\tilde{\nu}(s)$ are unique roots of the equations $f(a(s,z))=z$ and $\widetilde{f}(a_1(s,z))=z$ respectively on the interval $[0;1]$.

The sequences $q_i(s)$, $\tilde{q}_i(s)$ $(i\ge 0)$ are given in the form

$$q_i(s)=\int_0^\infty e^{-sx}\mathbf{P}\{\eta(x,\lambda)=i\}\overline{F}(x)dx=\sum_{k=0}^{i}a_i^{k*}\int_0^\infty e^{-(\lambda+s)x}\frac{(\lambda x)^k}{k!}\overline{F}(x)dx;$$

$$\tilde{q}_i(s)=\int_0^\infty e^{-sx}\mathbf{P}\{\eta(x,\lambda_1)=i\}\overline{\widetilde{F}}(x)dx=\sum_{k=0}^{i}a_i^{k*}\int_0^\infty e^{-(\lambda_1+s)x}\frac{(\lambda_1 x)^k}{k!}\overline{\widetilde{F}}(x)dx. \tag{3}$$

Introducing the notations

$$p_i = \lim_{s\to+0} p_i(s), \qquad R_i = \lim_{s\to+0} R_i(s), \qquad q_i = \lim_{s\to+0} q_i(s),$$
$$\tilde{p}_i = \lim_{s\to+0} \tilde{p}_i(s), \qquad \widetilde{R}_i = \lim_{s\to+0} \widetilde{R}_i(s), \qquad \tilde{q}_i = \lim_{s\to+0} \tilde{q}_i(s), \tag{4}$$

using (1)-(4) we obtain the relations:

$$p_i = \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} dF(x), \quad \tilde{p}_i = \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^\infty e^{-\lambda_1 x} \frac{(\lambda_1 x)^k}{k!} d\widetilde{F}(x), \quad i \geq -1;$$
$$R_1 = \frac{1}{p_{-1}}, \qquad R_{k+1} = \frac{1}{p_{-1}}\left(R_k - \sum_{i=0}^{k-1} p_i R_{k-i} \right), \qquad k \geq 1;$$
$$\widetilde{R}_1 = \frac{1}{\tilde{p}_{-1}}, \qquad \widetilde{R}_{k+1} = \frac{1}{\tilde{p}_{-1}}\left(\widetilde{R}_k - \sum_{i=0}^{k-1} \tilde{p}_i \widetilde{R}_{k-i} \right), \qquad k \geq 1; \tag{5}$$
$$q_0 = \frac{1 - f(\lambda)}{\lambda}, \qquad q_k = \sum_{i=1}^{k} a_i q_{k-i} - \frac{p_{k-1}}{\lambda}, \qquad k \geq 1;$$
$$\tilde{q}_0 = \frac{1 - \widetilde{f}(\lambda_1)}{\lambda_1}, \qquad \tilde{q}_k = \sum_{i=1}^{k} a_i \tilde{q}_{k-i} - \frac{\tilde{p}_{k-1}}{\lambda_1}, \qquad k \geq 1.$$

3. Distribution of the number of customers in the system during the busy period

Let $\tau(m) = \inf\{t \geq 0 : \xi(t) = 0\}$ denote the first busy period for the considered system and

$$\varphi_n(t,k) = \mathbf{P}_n\{\xi(t) = k, \tau(m) > t\}, \qquad 1 \leq n, k \leq m+1,$$
$$\Phi_n(s,k) = \int_0^\infty e^{-st} \varphi_n(t,k)\,dt, \qquad \operatorname{Re} s > 0.$$

We introduce the notations:

$$g_n(s,k) = g(s,k)\overline{p}_{m-n}(s) + q_{k-n}(s) + I\{k = m+1\}\overline{q}_{m-n+2}(s),$$
$$\tilde{g}_n(s,k) = \tilde{g}(s,k)\overline{\overline{p}}_{m-n}(s) + \tilde{q}_{k-n}(s) + I\{k = m+1\}\overline{\overline{q}}_{m-n+2}(s),$$
$$g(s,k) = I\{h+1 \leq k \leq m\} f(s) \widetilde{f}^{\,m-k}(s) \frac{1 - \widetilde{f}(s)}{s};$$
$$\tilde{g}(s,k) = I\{h+1 \leq k \leq m\} \widetilde{f}^{\,m+1-k}(s) \frac{1 - \widetilde{f}(s)}{s}.$$

Here $I\{A\}$ is the indicator of a random event A; it equals 1 or 0 depending on whether or not the event A occurs. Let also

$$L_n(s) = p_{h-n}(s) + \tilde{f}^{m-h}(s)\bar{p}_{m-n}(s) - \tilde{f}^{m+1-h}(s)\sum_{i=h+1-n}^{m-1-n} p_i(s)\sum_{j=1}^{m-n-i}\tilde{R}_j(s)\bar{\bar{p}}_{m-n-i-j}(s);$$

$$\Delta_h(s,k) = \sum_{i=1}^{h} R_i(s)\left(g_i(s,k) + f(s)\sum_{j=h+1-i}^{m-1-i} p_j(s)\left(\tilde{R}_{m-h}^{-1}(s)\tilde{R}_{m-i-j}(s) \times \right.\right.$$
$$\left.\left. \times\sum_{l=1}^{m-h}\tilde{R}_l(s)\tilde{g}_{h+l}(s,k) - \sum_{l=1}^{m-i-j}\tilde{R}_l(s)\tilde{g}_{i+j+l}(s,k)\right)\right);$$

$$\Delta(s) = R_h(s) - f(s)\sum_{i=1}^{h} R_i(s)L_i(s) - f(s)\tilde{R}_{m-h}^{-1}(s)\sum_{i=1}^{h} R_i(s)\times$$
$$\times\sum_{j=h+1-i}^{m-1-i} p_j(s)\tilde{R}_{m-i-j}(s)\left(1 + \tilde{f}^{m+1-h}(s)\sum_{i=1}^{m-h}\tilde{R}_i(s)\bar{\bar{p}}_{m-h-i}(s)\right).$$

Theorem 1. For $1 \le k \le m+1$ and $\operatorname{Re} s > 0$

$$\int_0^\infty e^{-st}\mathbf{P}_n\{\xi(t)=k,\ \tau(m)>t\}dt = \left(R_{h-n}(s) - f(s)\sum_{i=1}^{h-n} R_i(s)L_{n+i}(s)\right)\Phi_h(s,k) -$$
$$-\sum_{i=1}^{h-n} R_i(s)\left(f(s)\sum_{j=h+1-n-i}^{m-1-n-i} p_j(s)\left(\Phi_m(s,k)\tilde{R}_{m-n-i-j}(s) - \right.\right.$$
$$\left.\left. - \sum_{l=1}^{m-n-i-j}\tilde{R}_l(s)\tilde{g}_{n+i+j+l}(s,k)\right) + g_{n+i}(s,k)\right), \qquad 1 \le n \le h-1; \tag{6}$$

$$\int_0^\infty e^{-st}\mathbf{P}_n\{\xi(t)=k,\ \tau(m)>t\}dt = \tilde{R}_{m-n}(s)\Phi_m(s,k) - \tilde{f}^{m+1-h}(s)\Phi_h(s,k)\times$$
$$\times\sum_{i=1}^{m-n}\tilde{R}_i(s)\bar{\bar{p}}_{m-n-i}(s) - \sum_{i=1}^{m-n}\tilde{R}_i(s)\tilde{g}_{n+i}(s,k), \qquad h+1 \le n \le m-1,$$

where

$$\Phi_h(s,k) = \frac{\Delta_h(s,k)}{\Delta(s)}, \qquad \Phi_m(s,k) = \frac{1}{\tilde{R}_{m-h}(s)}\left(\left(1 + \tilde{f}^{m+1-h}(s)\times\right.\right.$$
$$\left.\left.\times\sum_{i=1}^{m-h}\tilde{R}_i(s)\bar{\bar{p}}_{m-h-i}(s)\right)\Phi_h(s,k) + \sum_{i=1}^{m-h}\tilde{R}_i(s)\tilde{g}_{h+i}(s,k)\right). \tag{7}$$

Proof. It is obvious that $\varphi_0(t,k)=0$. The total probability formula implies

$$\varphi_n(t,k)=\sum_{i=0}^{m-n}\int_0^t \mathbf{P}\{\eta(x,\lambda)=i\}\varphi_{n+i-1}(t-x,k)dF(x)+$$
$$+\int_0^t \mathbf{P}\{\eta(x,\lambda)\ge m+1-n\}\int_0^{t-x}\mathbf{P}\left\{\sum_{i=1}^{m-h}\widetilde{\beta}_i\in dv\right\}\varphi_h(t-x-v,k)dF(x)+$$
$$+I\{h+1\le k\le m\}\int_0^t \mathbf{P}\{\eta(x,\lambda))\ge m+1-n\}\mathbf{P}\left\{\sum_{i=1}^{m-k}\widetilde{\beta}_i<t-x\le\sum_{i=1}^{m+1-k}\widetilde{\beta}_i\right\}dF(x)+$$
$$+\left(\mathbf{P}\{\eta(t,\lambda))=k-n\}+I\{k=m+1\}\mathbf{P}\{\eta(t,\lambda)\ge m+2-n\}\right)\overline{F}(t),\qquad 1\le n\le h;$$

$$\varphi_n(t,k)=\sum_{i=0}^{m-n}\int_0^t \mathbf{P}\{\eta(x,\lambda_1)=i\}\varphi_{n+i-1}(t-x,k)d\widetilde{F}(x)+ \tag{8}$$

$$+\int_0^t \mathbf{P}\{\eta(x,\lambda_1)\ge m+1-n\}\int_0^{t-x}\mathbf{P}\left\{\sum_{i=1}^{m-h}\widetilde{\beta}_i\in dv\right\}\varphi_h(t-x-v,k)d\widetilde{F}(x)+$$
$$+I\{h+1\le k\le m\}\int_0^t \mathbf{P}\{\eta(x,\lambda_1)\ge m+1-n\}\mathbf{P}\left\{\sum_{i=1}^{m-k}\widetilde{\beta}_i<t-x\le\sum_{i=1}^{m+1-k}\widetilde{\beta}_i\right\}d\widetilde{F}(x)+$$
$$+\left(\mathbf{P}\{\eta(t,\lambda_1)=k-n\}+I\{k=m+1\}\mathbf{P}\{\eta(t,\lambda_1)\ge m+2-n\}\right)\overline{\overline{F}}(t),\quad h+1\le n\le m.$$

Passing to the Laplace transform on both sides of equalities (8) and taking into account relations (1)-(3), we obtain the system of equations with respect to the functions $\Phi_n(s,k)$ $(1\le n\le m)$

$$\begin{aligned}\Phi_n(s,k)&=f(s)\sum_{i=-1}^{m-n-1}p_i(s)\Phi_{n+i}(s,k)+f(s)\widetilde{f}^{\,m-h}(s)\overline{p}_{m-n}(s)\Phi_h(s,k)+\\&\quad+g_n(s,k),\qquad 1\le n\le h;\end{aligned} \tag{9}$$

$$\begin{aligned}\Phi_n(s,k)&=\widetilde{f}(s)\sum_{i=-1}^{m-n-1}\tilde{p}_i(s)\Phi_{n+i}(s,k)+\\&\quad+\widetilde{f}^{\,m+1-h}(s)\overline{\overline{p}}_{m-n}(s)\Phi_h(s,k)+\tilde{g}_n(s,k),\qquad h+1\le n\le m,\end{aligned} \tag{10}$$

and the boundary condition

$$\Phi_0(s,k)=0. \tag{11}$$

Using Lemma 1 of the paper [6], solutions of the system (10) are given by

$$\Phi_n(s,k) = \widetilde{R}_{m-n}(s)\Phi_m(s,k) - \widetilde{f}^{\,m+1-h}(s)\Phi_h(s,k)\sum_{i=1}^{m-n}\widetilde{R}_i(s)\bar{\tilde{p}}_{m-n-i}(s) - \\ -\sum_{i=1}^{m-n}\widetilde{R}_i(s)\tilde{g}_{n+i}(s,k), \qquad h \le n \le m-1. \tag{12}$$

With the help of (12) the system of equations (9) is rewritten as

$$\Phi_n(s,k) = f(s)\sum_{i=-1}^{h-n-1} p_i(s)\Phi_{n+i}(s,k) + f(s)L_n(s)\Phi_h(s,k) + f(s)\sum_{i=h+1-n}^{m-1-n} p_i(s)\times \\ \times\left(\Phi_m(s,k)\widetilde{R}_{m-n-i}(s) - \sum_{j=1}^{m-n-i}\widetilde{R}_j(s)\tilde{g}_{n+i+j}(s,k)\right) + g_n(s,k), \qquad 1 \le n \le h.$$

We again use Lemma 1 of [6] and deduce that

$$\Phi_n(s,k) = \left(R_{h-n}(s) - f(s)\sum_{i=1}^{h-n}R_i(s)L_{n+i}(s)\right)\Phi_h(s,k) - \\ -\sum_{i=1}^{h-n}R_i(s)\left(f(s)\sum_{j=h+1-n-i}^{m-1-n-i}p_j(s)\times\right. \\ \left.\times\left(\Phi_m(s,k)\widetilde{R}_{m-n-i-j}(s) - \sum_{l=1}^{m-n-i-j}\widetilde{R}_l(s)\tilde{g}_{n+i+j+l}(s,k)\right) + g_{n+i}(s,k)\right), \qquad 0 \le n \le h-1. \tag{13}$$

Considering the equality (12) with $n = h$ and (13) with $n = 0$, and using the condition (11), we obtain the system of two linear equations with respect to the functions $\Phi_h(s,k)$ and $\Phi_m(s,k)$:

$$\left(1 + \widetilde{f}^{\,m+1-h}(s)\sum_{i=1}^{m-h}\widetilde{R}_i(s)\bar{\tilde{p}}_{m-h-i}(s)\right)\Phi_h(s,k) - \widetilde{R}_{m-h}(s)\Phi_m(s,k) = -\sum_{i=1}^{m-h}\widetilde{R}_i(s)\tilde{g}_{h+i}(s,k);$$

$$\left(R_h(s) - f(s)\sum_{i=1}^{h}R_i(s)L_i(s)\right)\Phi_h(s,k) - f(s)\Phi_m(s,k)\sum_{i=1}^{h}R_i(s)\sum_{j=h+1-i}^{m-1-i}p_j(s)\widetilde{R}_{m-i-j}(s) = \\ = -f(s)\sum_{i=1}^{h}R_i(s)\sum_{j=h+1-i}^{m-1-i}p_j(s)\sum_{l=1}^{m-i-j}\widetilde{R}_l(s)\tilde{g}_{i+j+l}(s,k) + \sum_{i=1}^{h}R_i(s)g_i(s,k).$$

The solutions of this system are defined in form (7). Equalities (6) follow from the relations (12) and (13). The theorem is proved.

4. Busy period and stationary distribution

If the system starts functioning at the moment when the first batch of customers arrives, then

$$\int_0^\infty e^{-st}\mathbf{P}\{\xi(t)=k,\ \tau(m)>t\}dt=\sum_{n=1}^{m}a_n\Phi_n(s,k)+\overline{a}_{m+1}\Phi_{m+1}(s,k). \tag{14}$$

The total probability formula implies

$$\varphi_{m+1}(t,k)=\int_0^t\int_0^{t-x}\mathbf{P}\left\{\sum_{i=1}^{m-h}\widetilde{\beta}_i\in dv\right\}\varphi_h(t-x-v,k)d\widetilde{F}(x)+$$
$$+I\{h+1\le k\le m\}\int_0^t\mathbf{P}\left\{\sum_{i=1}^{m-k}\widetilde{\beta}_i<t-x\le\sum_{i=1}^{m+1-k}\widetilde{\beta}_i\right\}d\widetilde{F}(x)+I\{k=m+1\}\overline{\widetilde{F}}(t).$$

The Laplace transform of $\varphi_{m+1}(t,k)$ is given by

$$\Phi_{m+1}(s,k)=\widetilde{f}^{m-h+1}(s)\Phi_h(s,k)+I\{h+1\le k\le m+1\}\widetilde{f}^{m-k+1}(s)\frac{1-\widetilde{f}(s)}{s}. \tag{15}$$

Using the relations (12), (13) and (15) we rewrite the equation (14) in the form

$$\int_0^\infty e^{-st}\mathbf{P}\{\xi(t)=k,\tau(m)>t\}dt=\left(\sum_{n=1}^{h-1}a_n(R_{h-n}(s)-f(s)\sum_{i=1}^{h-n}R_i(s)L_{n+i}(s)\right)+a_h-$$
$$-\widetilde{f}^{m+1-h}(s)\sum_{n=h+1}^{m-1}a_n\sum_{i=1}^{m-n}\widetilde{R}_i(s)\overline{\widetilde{p}}_{m-n-i}(s)+\overline{a}_{m+1}\widetilde{f}^{m+1-h}(s))\Phi_h(s,k)-$$
$$-A(s)\Phi_m(s,k)+\sum_{n=1}^{h-1}a_n\sum_{i=1}^{h-n}R_i(s)(f(s)\sum_{j=h+1-n-i}^{m-1-n-i}p_j(s)\times$$
$$\times\sum_{l=1}^{m-n-i-j}\widetilde{R}_l(s)\tilde{g}_{n+i+j+l}(s,k)-g_{n+i}(s,k))-\sum_{n=h+1}^{m-1}a_n\sum_{i=1}^{m-n}\widetilde{R}_i(s)\tilde{g}_{n+i}(s,k)+$$
$$+\overline{a}_{m+1}I\{h+1\le k\le m+1\}\widetilde{f}^{m-k+1}(s)\frac{1-\widetilde{f}(s)}{s}, \tag{16}$$

where

$$A(s)=f(s)\sum_{n=1}^{h-1}a_n\sum_{i=1}^{h-n}R_i(s)\sum_{j=h+1-n-i}^{m-1-n-i}p_j(s)\widetilde{R}_{m-n-i-j}(s)-\sum_{n=h+1}^{m-1}a_n\widetilde{R}_{m-n}(s)-a_m.$$

To obtain a representation for $\int_0^\infty e^{-st}\mathbf{P}\{\tau(m)>t\}dt$ we sum up equalities (16) for k running from 1 to $m+1$. Making sure that

$$g_n(s)=\sum_{k=1}^{m+1} g_n(s,k)=f(s)\frac{1-\tilde{f}^{m-h}(s)}{s}\bar{p}_{m-n}(s)+\frac{1-f(s)}{s};$$

$$\tilde{g}_n(s)=\sum_{k=1}^{m+1}\tilde{g}_n(s,k)=\tilde{f}(s)\frac{1-\tilde{f}^{m-h}(s)}{s}\bar{\tilde{p}}_{m-n}(s)+\frac{1-\tilde{f}(s)}{s};$$

$$\Delta_h(s)=\sum_{k=1}^{m+1}\Delta_h(s,k)=\sum_{i=1}^{h}R_i(s)\Bigg(g_i(s)+f(s)\sum_{j=h+1-i}^{m-1-i}p_j(s)\Big(\tilde{R}_{m-h}^{-1}(s)\tilde{R}_{m-i-j}(s)\times$$

$$\times\sum_{l=1}^{m-h}\tilde{R}_l(s)\tilde{g}_{h+l}(s)-\sum_{l=1}^{m-i-j}\tilde{R}_l(s)\tilde{g}_{i+j+l}(s)\Big)\Bigg);\qquad \Phi_h(s)=\sum_{k=1}^{m+1}\Phi_h(s,k)=\frac{\Delta_h(s)}{\Delta(s)};$$

$$\Phi_m(s)=\sum_{k=1}^{m+1}\Phi_m(s,k)=\frac{1}{\tilde{R}_{m-h}(s)}\Bigg(\Big(1+\tilde{f}^{m+1-h}(s)\sum_{i=1}^{m-h}\tilde{R}_i(s)\bar{\tilde{p}}_{m-h-i}(s)\Big)\Phi_h(s)+$$

$$+\sum_{i=1}^{m-h}\tilde{R}_i(s)\tilde{g}_{h+i}(s)\Bigg),$$

From (16) we obtain an expression for the Laplace transform of the distribution function of the busy period

$$\begin{aligned}\int_0^\infty e^{-st}\mathbf{P}\{\tau(m)>t\}dt=&\Bigg(\sum_{n=1}^{h-1}a_n\Big(R_{h-n}(s)-f(s)\sum_{i=1}^{h-n}R_i(s)L_{n+i}(s)\Big)+a_h-\\&-\tilde{f}^{m+1-h}(s)\sum_{n=h+1}^{m-1}a_n\sum_{i=1}^{m-n}\tilde{R}_i(s)\bar{\tilde{p}}_{m-n-i}(s)+\bar{a}_{m+1}\tilde{f}^{m+1-h}(s)\Bigg)\Phi_h(s)-\\&-A(s)\Phi_m(s)+\sum_{n=1}^{h-1}a_n\sum_{i=1}^{h-n}R_i(s)\Bigg(f(s)\sum_{j=h+1-n-i}^{m-1-n-i}p_j(s)\times\\&\times\sum_{l=1}^{m-n-i-j}\tilde{R}_l(s)\tilde{g}_{n+i+j+l}(s)-g_{n+i}(s)\Bigg)-\\&-\sum_{n=h+1}^{m-1}a_n\sum_{i=1}^{m-n}\tilde{R}_i(s)\tilde{g}_{n+i}(s)+\bar{a}_{m+1}\frac{1-\tilde{f}^{m+1-h}(s)}{s}.\end{aligned}\tag{17}$$

Passing to the limit in (17) as $s\to+0$, we derive the formula for the mean duration of a busy period of the studied queueing system. To calculate this limit using the sequences defined by (4), (5) and take into account the following limit relations:

$$\lim_{s\to+0}\frac{1-f(s)}{s}=M;\qquad \lim_{s\to+0}\frac{1-\tilde{f}^n(s)}{s}=n\widetilde{M}\quad (n\ge 1);\qquad \lim_{s\to+0}\Delta(s)=1;$$

$$\lim_{s\to+0}g_n(s)=M+(m-h)\widetilde{M}\overline{p}_{m-n};\qquad \lim_{s\to+0}\tilde{g}_n(s)=\widetilde{M}\left(1+(m-h)\overline{\tilde{p}}_{m-n}\right);$$

$$A=\lim_{s\to+0}A(s)=\sum_{n=1}^{h-1}a_n\sum_{i=1}^{h-n}R_i\sum_{j=h+1-n-i}^{m-1-n-i}p_j\widetilde{R}_{m-n-i-j}-\sum_{n=h+1}^{m-1}a_n\widetilde{R}_{m-n}-a_m.$$

Therefore we proved the following result.

Theorem 2. The mean duration of the busy period of the queueing system is determined in the form

$$\begin{aligned}\mathbf{E}\tau(m)=&\sum_{i=1}^{h}R_i\left(M+(m-h)\widetilde{M}\overline{p}_{m-i}+\widetilde{M}\widetilde{R}_{m-h}^{-1}\sum_{j=h+1-i}^{m-1-i}p_j\left(\widetilde{R}_{m-i-j}\sum_{l=1}^{m-h-1}\widetilde{R}_l-\right.\right.\\&\left.\left.-\widetilde{R}_{m-h}\sum_{l=1}^{m-i-j-1}\widetilde{R}_l+(m-h)\left(\widetilde{R}_{m-h}-\widetilde{R}_{m-i-j}\right)\right)\right)-\\&-C_hA\widetilde{R}_{m-h}^{-1}+\sum_{n=1}^{h-1}a_n\sum_{i=1}^{h-n}R_i\left(\sum_{j=h+1-n-i}^{m-1-n-i}p_jC_{n+i+j}-M-(m-h)\widetilde{M}\overline{p}_{m-n-i}\right)-\\&-\sum_{n=h+1}^{m-1}a_nC_n+(m+1-h)\widetilde{M}\widetilde{a}_{m+1},\end{aligned}\tag{18}$$

where

$$C_n=\widetilde{M}\left(\sum_{i=1}^{m-n}\widetilde{R}_i+(m-h)\left(\widetilde{R}_{m-n}-1\right)\right).$$

Applying the same reasoning as that in the paper [11, p. 169-170] and using the key renewal theorem [12, p. 46], we obtain the equalities

$$\begin{aligned}&\lim_{t\to\infty}\mathbf{P}\{\xi(t)=k\}=\frac{\lambda}{1+\lambda\mathbf{E}\tau(m)}\int_0^\infty\mathbf{P}\{\xi(u)=k,\tau(m)\ge u\}du,\qquad 1\le k\le m+1;\\&\lim_{t\to\infty}\mathbf{P}\{\xi(t)=0\}=\frac{\lambda}{1+\lambda\mathbf{E}\tau(m)}\int_0^\infty\mathbf{P}\{\tau(m)<u,\tau(m)+\xi_1\ge u\}du.\end{aligned}\tag{19}$$

Since $\mathbf{P}\{\tau(m)<u,\tau(m)+\xi_1\ge u\}=\mathbf{P}\{\tau(m)+\xi_1\ge u\}-\mathbf{P}\{\tau(m)\ge u\}$, then

$$\int_0^\infty\mathbf{P}\{\tau(m)<u,\tau(m)+\xi_1\ge u\}du=\frac{1}{\lambda}.\tag{20}$$

Passing in (16) to the limit as $s \to +0$ we get

$$\int_0^\infty \mathbf{P}\{\xi(t)=k,\tau(m)>t\}dt = \delta_h(k) - A\widetilde{R}_{m-h}^{-1}\sum_{i=1}^{m-h}\widetilde{R}_i\widetilde{G}_{h+i}(k) + \sum_{n=1}^{h-1}a_n\sum_{i=1}^{h-n}R_i \times$$

$$\times\left(\sum_{j=h+1-n-i}^{m-1-n-i}p_j\sum_{l=1}^{m-n-i-j}\widetilde{R}_l\widetilde{G}_{n+i+j+l}(k) - G_{n+i}(k)\right) - \sum_{n=h+1}^{m-1}a_n\sum_{i=1}^{m-n}\widetilde{R}_i\widetilde{G}_{n+i}(k) + \qquad (21)$$

$$+\widetilde{M}\overline{a}_{m+1}I\{h+1\le k\le m+1\},$$

where

$$\delta_h(k) = \sum_{i=1}^{h}R_i\left(G_i(k) + \sum_{j=h+1-i}^{m-1-i}p_j\left(\widetilde{R}_{m-h}^{-1}\widetilde{R}_{m-i-j}\sum_{l=1}^{m-h}\widetilde{R}_l\widetilde{G}_{h+l}(k) - \sum_{l=1}^{m-i-j}\widetilde{R}_l\widetilde{G}_{i+j+l}(k)\right)\right);$$

$$G_i(k) = q_{k-i} + I\{h+1\le k\le m\}\widetilde{M}\overline{p}_{m-i} + I\{k=m+1\}\overline{q}_{m+2-i};$$

$$\widetilde{G}_i(k) = \tilde{q}_{k-i} + I\{h+1\le k\le m\}\widetilde{M}\overline{\tilde{p}}_{m-i} + I\{k=m+1\}\overline{\tilde{q}}_{m+2-i}.$$

Introducing the notations: $\rho_k(m) = \lim_{t\to\infty}\mathbf{P}\{\xi(t)=k\}, \quad 0\le k\le m+1;$

$$B_n(k) = \sum_{i=1}^{h-n}R_iG_{n+i}(k) = \sum_{i=1}^{h-n}R_iq_{k-n-i} + \widetilde{M}\sum_{i=1}^{h-n}R_i\overline{p}_{m-n-i};$$

$$D_n(k) = \sum_{i=1}^{m-n}\widetilde{R}_i\widetilde{G}_{n+i}(k) = \sum_{i=1}^{k-n}\widetilde{R}_i\tilde{q}_{k-n-i} + \widetilde{M}\left(\widetilde{R}_{m-n}-1\right).$$

Taking into account (19), (20) and equality $\sum_{i=1}^{n}\widetilde{R}_i\overline{\tilde{p}}_{n-i} = \widetilde{R}_n - 1$, which follows from (5), with the help of (21) we obtain the following statement.

Theorem 3. The stationary distribution of the number of customers in the system is given by

$$\rho_0(m) = \frac{1}{1+\lambda\mathbf{E}\tau(m)};$$

$$\rho_k(m) = \lambda\rho_0(m)\left(\sum_{i=1}^{k}R_iq_{k-i} - \sum_{n=1}^{k-1}a_n\sum_{i=1}^{k-n}R_iq_{k-n-i}\right), \qquad 1\le k\le h; \qquad (22)$$

$$-\sum_{l=1}^{m-i-j}\widetilde{R}_l\overline{\tilde{q}}_{m+1-i-j-l}\Bigg)\Bigg) - \sum_{n=h+1}^{m-1}a_n\sum_{i=1}^{m-n}\widetilde{R}_i\overline{\tilde{q}}_{m+1-n-i} - A\widetilde{R}_{m-h}^{-1}\sum_{i=1}^{m-h}\widetilde{R}_i\overline{\tilde{q}}_{m+1-h-i} + \widetilde{M}\overline{a}_{m+1} +$$

$$+\sum_{n=1}^{h-1}a_n\sum_{i=1}^{h-n}R_i\left(\sum_{j=h+1-n-i}^{m-1-n-i}p_j\sum_{l=1}^{m-n-i-j}\widetilde{R}_l\overline{\tilde{q}}_{m+1-n-i-j-l} - \overline{q}_{m+1-n-i}\right)\Bigg).$$

$$\rho_k(m) = \lambda\rho_0(m)\Bigg(B_0(k) - \sum_{n=1}^{h-1} a_n B_n(k) + \sum_{i=1}^{h} R_i \sum_{j=h+1-i}^{m-1-i} p_j \Big(\widetilde{R}_{m-h}^{-1} \widetilde{R}_{m-i-j} D_h(k) -$$

$$- D_{i+j}(k)\Big) - A\widetilde{R}_{m-h}^{-1} D_h(k) + \sum_{n=1}^{h-1} a_n \sum_{i=1}^{h-n} R_i \sum_{j=h+1-n-i}^{m-1-n-i} p_j D_{n+i+j}(k) -$$

$$- \sum_{n=h+1}^{m-1} a_n D_n(k) + \widetilde{M}\overline{a}_{m+1} \Bigg), \qquad h+1 \le k \le m;$$

$$\rho_{m+1}(m) = \lambda\rho_0(m)\Bigg(\sum_{i=1}^{h} R_i \Bigg(\overline{q}_{m+1-i} + \sum_{j=h+1-i}^{m-1-i} p_j \Bigg(\widetilde{R}_{m-h}^{-1} \widetilde{R}_{m-i-j} \sum_{l=1}^{m-h} \widetilde{R}_l \overline{\widetilde{q}}_{m+1-h-l} -$$

$$- \sum_{l=1}^{m-i-j} \widetilde{R}_l \overline{\widetilde{q}}_{m+1-i-j-l} \Bigg)\Bigg) - \sum_{n=h+1}^{m-1} a_n \sum_{i=1}^{m-n} \widetilde{R}_i \overline{\widetilde{q}}_{m+1-n-i} - A\widetilde{R}_{m-h}^{-1} \sum_{i=1}^{m-h} \widetilde{R}_i \overline{\widetilde{q}}_{m+1-h-i} + \widetilde{M}\overline{a}_{m+1} +$$

$$+ \sum_{n=1}^{h-1} a_n \sum_{i=1}^{h-n} R_i \Bigg(\sum_{j=h+1-n-i}^{m-1-n-i} p_j \sum_{l=1}^{m-n-i-j} \widetilde{R}_l \overline{\widetilde{q}}_{m+1-n-i-j-l} - \overline{q}_{m+1-n-i} \Bigg)\Bigg).$$

5. Determination of stationary characteristics

Using the equality (18), the expression for the mean duration of the busy period can be written as $\mathbf{E}\tau(m) = MT(m) + \widetilde{M}\widetilde{T}(m)$. Then for the mean service time we obtain the formula

$$\overline{M} = \frac{\mathbf{E}\tau(m)}{T(m) + \widetilde{T}(m)}.$$

The formula for the probability of service $\mathbf{P}_{\mathrm{sv}}(m)$ we get as a ratio of the mean number of served customers by the mean number of arrivals per unit of time

$$\mathbf{P}_{\mathrm{sv}}(m) = \frac{\big(1 - \rho_0(m)\big)\big(T(m) + \widetilde{T}(m)\big)}{\lambda e_a \mathbf{E}\tau(m)} = \frac{\rho_0(m)\big(T(m) + \widetilde{T}(m)\big)}{e_a}. \tag{23}$$

Stationary queue characteristics, namely the mean queue length $\mathbf{E}Q(m)$ and mean waiting time of service $\mathbf{E}w(m)$, we can find by the formulas

$$\mathbf{E}Q(m) = \sum_{k=1}^{m} k\rho_{k+1}(m); \qquad \mathbf{E}w(m) = \frac{\mathbf{E}Q(m)}{\lambda e_a \mathbf{P}_{\mathrm{sv}}(m)}. \tag{24}$$

6. Example of calculation of the stationary distribution and the characteristics of the system

Assume that $m = 9$, $h = 5$, customers can arrive only one by one or two by two ($a_1 = 0.75$, $a_2 = 0.25$), $\lambda_1 = \lambda_2 = 1$, in normal mode the service time is uniformly distributed on the interval $[1/3; 1]$ with the mean value $M = 2/3$, and in partial and full blocking modes the service time is uniformly distributed on the interval $[0; 2/3]$ with the mean value $\widetilde{M} = 1/3$. Then, the mean duration of the busy period $\mathbf{E}\tau(m)$ found by the formula (18) is equal to 118.319. The second row of Table 1 contains the probabilities $\rho_k(m)$, calculated by formulas (22). For the sake of comparison, the same table contains the corresponding probabilities evaluated by the GPSS World simulation system [13] for the time value $t = 10^6$. The values of the stationary characteristics, found by formulas (23) and (24) and calculated with the help of GPSS World, are shown in Table 2.

Table 1

Stationary distributions of the number of customers in the system

Number of customers (k)	0	1	2	3	4	5
$\rho_k(m)$ according to (22)	0.00421	0.01063	0.02426	0.05302	0.11529	0.25042
$\rho_k(m)$ (GPSS World)	0.00429	0.01094	0.02447	0.05336	0.11565	0.25041
Number of customers (k)	6	7	8	9	10	
$\rho_k(m)$ according to (22)	0.23442	0.16229	0.09027	0.04274	0.01245	
$\rho_k(m)$ (GPSS World)	0.23403	0.16228	0.08958	0.04240	0.01259	

Table 2

Stationary system characteristics

Characteristic	$\mathbf{P}_{sv}(m)$	$\mathbf{E}Q(m)$	$\mathbf{E}w(m)$
Analytical value	0.78721	4.70955	2.39302
Value according to GPSS World	0.787	4.704	2.391

Conclusions

The advantage of the proposed algorithm for calculating the stationary characteristics is that the recurrence relations (5), used to define the sequences $\{p_k\}$, $\{\tilde{p}_k\}$, $\{q_k\}$, $\{\tilde{q}_k\}$, $\{R_k\}$, $\{\widetilde{R}_k\}$, do not depend explicitly on the volume of storage heater m and the threshold value h and depend only on the parameters of the input flow and the distribution functions $F(x)$ and $\widetilde{F}(x)$ of the service time.

Therefore, in case of a change of parameter *m* and *h* there is no need to recalculate the values of these sequences.

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