SOLUTION OF DIFFERENTIAL EQUATION FOR THE EULER-BERNOULLI BEAM

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Abstract. The paper presents the solution of a fourth order differential equation with various coefficients occurring in the vibration problem of the Euler-Bernoulli beam. The concerning equation is written as a first order matrix differential equation. To solve the equation, the power series method is proposed.

Keywords: *Euler-Bernoulli beam, power series method, mathematical modelling*

Introduction

For certain cases of differential equations with variable coefficients, it is possible to determine their exact solutions [1-4], using e.g. homotopy analysis [3] or the Green's functions method [4]. However, in most cases, in order to obtain a solution it is necessary to apply approximate methods, such as finite difference method [5], the power series method [6], differential transformation method (DTM) - "improved" Taylor method [5, 7, 8] or by the use of the Lagrange multiplier formalism [9]. This paper is a continuation of consideration, shown at [10], relating to the use of a matrix and power series methods for solving ordinary differential equations.

The work presents, as an example for proposed method, the solution to the equation of motion of the non-uniform beam, described according to the Euler-Bernoulli theory, by the equation of the fourth order.

Formulation and solution of the problem

Basic concepts of the procedure

At the beginning, let us recall the procedure schema for a fourth order linear differential equation:

$$y^{(4)} + a_4(x)y^{(3)} + a_3(x)y'' + a_2(x)y' + a_1(x)y = f(x) \tag{1}$$

completed by initial conditions. By introducing functions: $y(x)=y_1(x)$, $y_1'(x)=y_2(x)$, $y_2'(x)=y_3(x)$, $y_3'(x)=y_4(x)$ and $y_4'(x)=f(x)-\sum_{s=1}^{4}a_s(x)y_s(x)$, assuming that the functions appearing in the equation (1) are C^∞ class and can be presented in the form of power series, we can rewrite (1) as a first order matrix differential equation [10]:

$$\mathbf{Y}'(x)=\mathbf{B}(x)\mathbf{Y}(x)+\mathbf{F}(x) \tag{2}$$

In the above equation, the following designations were charged:

$$\mathbf{Y}(x)=\left[y_1(x)\ y_2(x)\ y_3(x)\ y_4(x)\right]^T=\sum_{n=0}^{\infty}\left[y_{1n}\ y_{2n}\ y_{3n}\ y_{4n}\right]^T\frac{x^n}{n!}=\sum_{n=0}^{\infty}\mathbf{Y}_n\frac{x^n}{n!}$$

$$\mathbf{F}(x)=\left[0\ 0\ 0\ f(x)\right]^T=\sum_{n=0}^{\infty}\left[0\ 0\ 0\ f_n\right]^T\frac{x^n}{n!}=\sum_{n=0}^{\infty}\mathbf{F}_n\frac{x^n}{n!},$$

$$\mathbf{B}(x)=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -a_1(x) & -a_2(x) & -a_3(x) & -a_4(x)\end{bmatrix}=\sum_{n=0}^{\infty}\mathbf{B}_n\frac{x^n}{n!},$$

$$\mathbf{B}_0=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -a_{10} & -a_{20} & -a_{30} & -a_{40}\end{bmatrix},\ \mathbf{B}_n=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -a_{1n} & -a_{2n} & -a_{3n} & -a_{4n}\end{bmatrix}\ \text{for } n\ge 1$$

and a boundary condition is $\mathbf{Y}(0)=\mathbf{Y}_0$.

A solution of an inhomogeneous equation (2), in the power series form, can be expressed as a sum

$$\mathbf{Y}(x)=\sum_{n=0}^{\infty}\mathbf{Y}_n(x)=\sum_{n=0}^{\infty}\left(\mathbf{\Phi}_n+\mathbf{\Psi}_n\mathbf{Y}_0\right)\frac{x^n}{n!} \tag{3}$$

where $\mathbf{\Phi}_0=0$, $\mathbf{\Psi}_0=\mathbf{E}$ and $\mathbf{\Phi}_n$, $\mathbf{\Psi}_n$ are determined from the recursive relations. The first few values of those parameters are:

$$\begin{aligned}
&\mathbf{\Phi}_1=\mathbf{F}_0,\ \mathbf{\Phi}_2=\mathbf{F}_1+\mathbf{B}_0\mathbf{F}_0,\ \mathbf{\Phi}_3=\mathbf{F}_2+\mathbf{B}_0\mathbf{F}_1+\left(\mathbf{B}_0^2+2\mathbf{B}_1\right)\mathbf{F}_0,\\
&\mathbf{\Phi}_4=\mathbf{F}_3+\mathbf{B}_0\mathbf{F}_2+\left(\mathbf{B}_0^2+3\mathbf{B}_1\right)\mathbf{F}_1+\left(\mathbf{B}_0^3+2\mathbf{B}_0\mathbf{B}_1+3\mathbf{B}_1\mathbf{B}_0+3\mathbf{B}_1\mathbf{B}_0^2+3\mathbf{B}_1^2+3\mathbf{B}_2\right)\mathbf{F}_0\\
&\mathbf{\Psi}_1=\mathbf{B}_0,\ \mathbf{\Psi}_2=\mathbf{B}_0^2+\mathbf{B}_1,\ \mathbf{\Psi}_3=\mathbf{B}_0^3+2\mathbf{B}_0\mathbf{B}_1+3\mathbf{B}_1\mathbf{B}_0+3\mathbf{B}_1\mathbf{B}_0^2+3\mathbf{B}_1^2+3\mathbf{B}_2\\
&\mathbf{\Psi}_4=\mathbf{B}_0^4+\mathbf{B}_0^2\mathbf{B}_1+2\mathbf{B}_0\mathbf{B}_1\mathbf{B}_0+\mathbf{B}_0\mathbf{B}_2+3\mathbf{B}_1\mathbf{B}_0^2+3\mathbf{B}_1^2+3\mathbf{B}_2\mathbf{B}_0+\mathbf{B}_3
\end{aligned} \tag{4}$$

Beam's equation of motion

According to the Euler-Bernoulli theory, the motion of the beam length L, with a various cross-section area $A(x)$ and moment of inertia $I(x)$ (Fig. 1), is described by the partial differential equation:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right]+\rho A(x)\frac{\partial^2 u}{\partial t^2}=0 \tag{5}$$

where u is the function of deflection, ρ is the mass density and E is the Young's modulus. Equation (5) is complemented by appropriate boundary conditions depending on the method of fixing the ends of the beam.

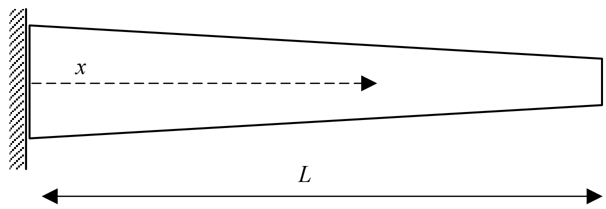

Fig. 1. A sketch of considered beam

Assuming a sinusoidal rotation of function $u(x,t)=y(x)e^{i\omega t}$, the equation of motion can be rewritten as:

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2 y}{dx^2}\right]-\rho\omega^2 A(x)y=0 \tag{6}$$

with natural frequency ω. Next, let us assume the cross-section area and the moment of inertia in the form of polynomials:

$$A(x)=A_0\left(\frac{\alpha-1}{L}x+1\right)^2,\ I(x)=I_0\left(\frac{\alpha-1}{L}x+1\right)^4 \tag{7}$$

where $A_0=A(0)$, $I_0=I(0)$, $\alpha\neq 1$ is a proportionality factor of the beam's cross-section and $x\in[0,L]$. After a few transformations, equation (6) can be expressed as equation (1), where:

$$a_1(x)=-\Omega^4 a^2(x),\ a_2(x)\equiv 0,\ a_3(x)=\frac{12(\alpha-1)}{L}a^2(x),\ a_4(x)=\frac{8(\alpha-1)}{L}a(x) \tag{8}$$

for $a(x)=\left(\dfrac{\alpha-1}{L}x+1\right)^{-1}$.

Parameter Ω characterizes the vibration frequency of the beam and is given by formula $\Omega^4 = \frac{\omega^2 \rho A_0}{EI_0}$. Note that functions $a(x)$ and $a^2(x)$ can be presented as power series:

$$a(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!},\ a^2(x) = \sum_{n=0}^{\infty} (n+1) a_n \frac{x^n}{n!} \quad \text{where } a_n = \left(\frac{1-\alpha}{L}\right)^n \cdot n! \tag{9}$$

Of course, we have to ensure the convergence of the series, hence:
for $\alpha > 1$ is $x \in \left\langle 0, \frac{L}{\alpha - 1} \right\rangle$ and for $\alpha < 1$ is $x \in \left\langle 0, \frac{L}{1-\alpha} \right\rangle$.
Matrix $\mathbf{B}(x)$ occurring in (2) is in the form:

$$\mathbf{B}(x) = \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega^4 (n+1) a_n & 0 & \frac{12(1-\alpha)}{L}(n+1) a_n & \frac{8(1-\alpha)}{L} a_n \end{bmatrix} \cdot \frac{x^n}{n!} \tag{10}$$

and solution of (6) is given by relation (3) for $\mathbf{\Phi}_n = 0$ and $\mathbf{\Psi}_n$ as in (4).

In the case $\alpha = 1$ we've got a vibration problem of the Euler-Bernoulli beam with constant parameters characterizing its physical properties. Equation (5) has the form:

$$y^{(4)} - \Omega^4 y = 0 \tag{11}$$

We can note, that $a_4(x) = a_3(x) = a_2(x) \equiv 0,\ a_1(x) = -\Omega^4$. After conversion, according to discussed method, matrices $\mathbf{B}_n$ are as follows:

$$\mathbf{B}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega^4 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B}_n = \mathbf{O},\ n \ge 1 \tag{12}$$

The solution of the boundary problem: $\mathbf{Y}'(x) = \mathbf{B}(x)\mathbf{Y}(x) + \mathbf{F}(x),\ \ \mathbf{Y}(0) = \mathbf{Y}_0$ is $\mathbf{Y}(x) = \sum_{n=0}^{\infty} \mathbf{B}_0^n \mathbf{Y}_0$, where:

$$\mathbf{B}_0^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega^4 & 0 & 0 & 0 \\ 0 & \Omega^4 & 0 & 0 \end{bmatrix}, \ \mathbf{B}_0^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \Omega^4 & 0 & 0 & 0 \\ 0 & \Omega^4 & 0 & 0 \\ 0 & 0 & \Omega^4 & 0 \end{bmatrix}, \ \mathbf{B}_0^4 = \Omega^4 \mathbf{E}$$

$$\mathbf{B}_0^{4k+1} = \left(\Omega^4\right)^k \mathbf{B}_0, \ \mathbf{B}_0^{4k+2} = \left(\Omega^4\right)^k \mathbf{B}_0^2, \tag{13}$$
$$\mathbf{B}_0^{4k+3} = \left(\Omega^4\right)^k \mathbf{B}_0^3, \ \mathbf{B}_0^{4k+1} = \left(\Omega^4\right)^k \mathbf{B}_0^4 = \left(\Omega^4\right)^{k+1} \mathbf{E}, \ k = 1,2,3,...$$

The general solution of equation (6) depends on four constants, which are determined from the boundary conditions. For example, the boundary conditions for a cantilever beam are

$$y(0) = y'(0) = 0, \ y''(L) = y'''(L) = 0 \tag{14}$$

The conditions at the beam's boundary $x = 0$ gives us $y_{10} = 0, y_{20} = 0$ and at $x = L$ gives $y_{30} = -\sum_{n=1}^{\infty} y_{3n} \frac{L^n}{n!}$, $y_{40} = -\sum_{n=1}^{\infty} y_{4n} \frac{L^n}{n!}$, so the initial condition to the (2) is

$$\mathbf{Y}_0 = \left[0 \ 0 \ -y_{30} \ -y_{40}\right]^T \tag{15}$$

Conclusions

In the present study, the method reducing the fourth order differential equation to the form of the first order matrix equation was considered on the example of non-uniform beam's vibration equation. Considerations show that it is a relatively simple method of solving the boundary problem, reduced to a form which is easy for computer implementation. If only the functions occurring in the equations are the appropriate class, and it's possible to expand them in power series, then in the interval of convergence of these series, the presented method can also be used to solve more complex problems described by differential equations.

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