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RECURRENCE RELATIONS FOR TWO-CHANNEL QUEUEING SYSTEMS WITH ERLANGIAN SERVICE TIMES AND HYSTERETIC STRATEGY OF RANDOM DROPPING OF CUSTOMERS

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Abstract. This article proposes a method of study the $M/E_s/2/m$ and $M/E_s/2/\infty$ queueing systems with a hysteretic strategy of random dropping of customers. Recurrence relations are obtained to compute the stationary distribution of the number of customers and steady-state characteristics. The constructed algorithms were tested on examples with the use of simulation models constructed with the help of GPSS World.

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1. Introduction

For investigating queueing systems with Erlang distributions and, in particular, the $M/E_s/n/\infty$ system, the fictitious phase method developed by A.K. Erlang [1] is used. For the Erlang distribution of the sth order of service time, it is supposed that each customer sequentially passes through s phases of service whose durations are distributed by exponential laws with parameters $\mu_1, \mu_2, \ldots, \mu_s$, respectively.

Accounting for phases requires the fixation of the corresponding states and leads to the increase in the cumbersomeness of the description of a queuing system with phase-type distributions. The direct solution of a system of equations for steady-state probabilities of states can result in being impossible in view of a large size of the coefficient matrix of a system. The algorithmic approach is most expedient since it presumes the obtaining of a solution to systems of equations in the form of recursive formulas or in the form of matrix recurrence relations and algorithms [2-10]. The method proposed in [7-10] is based on the use of direct recurrence relations following immediately from system equations for steady-state probabilities. It does not contain iterations and it does not presume preliminary transformations of the system of equations being solved.

The objective of this article is the construction of recurrent algorithms with the help of the fictitious phase method to compute the steady-state distribution of the number of customers in the $M/E_s/2/m$ and $M/E_s/2/\infty$ queuing system, where $s \geq 2$, with hysteretic strategy of the random dropping of customers. The random dropping of customers is used in queuing systems with a view for preventing overloads when each arriving customer can be discarded with a definite probability dependent on the queue length at the moment of arrival of a customer even if the buffer is not completely filled [11-13].

2. The $M/E_s/2/m$ system with hysteretic strategy of random dropping of customers

Let us consider the $M/E_s/2/m$ system, where $s \geq 2$ and m is the maximum number of customers who can simultaneously be in the queue. The input flow of customers is Poisson, i.e., the time intervals between the moments of arrival of customers adjacent in time are independent random variables exponentially distributed with the parameter λ. The service time of each customer is distributed according to the generalized Erlang law, of the order s, so the service time is the sum of s independent random variables exponentially distributed with parameters $\mu_1, \mu_2, \ldots, \mu_s$ respectively.

We consider the random dropping of customers that is implemented according to the following rule: if, at the moment of arrival of a customer, the number of customers in the system is equal to k (without making allowance for the arrived one), then the customer is accepted for service with probability β_k $(0 < \beta_k \leq 1$, $\beta_{m+2} = 0)$ and is refused (discarded) with probability $1 - \beta_k$.

We consider a hysteretic strategy of the random dropping of customers with two thresholds h_1 and h_2 $(3 \leq h_1 < h_2 < m+1)$ and with two operating modes, namely, basic and dropping mode. Assume that $\beta_k = 1$ when $0 \leq k \leq h_2 - 1$ for the basic mode and $0 < \beta_k < 1$ when $h_1 + 1 \leq k \leq m+1$ for the dropping mode. The dropping mode continues from the moment when the number of customers in the system achieves the value of h_2 up to the moment when the number of customers is reduced to the value of h_1. If, at the moment of arrival of a customer, the condition $h_1 < k < h_2$ is satisfied, then the mode is not changed. The rate of the simplest flow of customers accepted for service in the dropping mode is equal to $\lambda_k = \lambda\beta_k$. In a partial case of the hysteretic strategy, assuming that $\beta_k = \beta$ $(0 < \beta < 1)$ for all k in the dropping mode. If $m = \infty$, then $\beta_k = \tilde{\beta}$, $k \geq h \geq h_2$, where $0 < \tilde{\beta} < 1$.

The $M/E_s/2/m$ system with the hysteretic strategy of the random dropping of customers we denote by $M(h_1,h_2)/E_s/2/m$. We introduce the following designations for system states in the basic mode: s_0 signifies that customers are absent in the system; $s_{k(ij)}$ signifies that k customers are present in the system $(1 \le k \le h_2 - 1)$, and that two customers are at the ith phase of service and at the jth one $(1 \le i \le s, i \le j \le s)$ respectively. The states $s_{1(0j)}$ $(1 \le j \le s)$ correspond to one working channel and jth phase of service. We denote the steady-state probabilities of staying the system at the states s_0 and $s_{k(ij)}$ by p_0 and $p_{k(ij)}$, respectively. Let $\tilde{s}_{k(ij)}$ be the state similar to $s_{k(ij)}$ in the dropping mode and $q_{k(ij)}$ be the steady-state probability of staying the system at the state $\tilde{s}_{k(ij)}$ $(h_1 + 1 \le k \le\le m + 2,$ $1 \le i \le s, i \le j \le s)$. We assume that $p_{1(0j)} = p_{1(1j)}$ $(1 \le j \le s)$.

To determine steady-state probabilities, we obtain the system of homogeneous algebraic equations with normalization condition

$$p_0 + \sum_{j=1}^{s} p_{1(0j)} + \sum_{k=2}^{h_2-1} \sum_{i=1}^{s} \sum_{j=i}^{s} p_{k(ij)} + \sum_{k=h_1+1}^{m+2} \sum_{i=1}^{s} \sum_{j=i}^{s} q_{k(ij)} = 1. \qquad (1)$$

Introducing the notation

$$\alpha_i = \frac{\lambda}{\mu_i},\ 1 \le i \le s; \quad \eta_i = \frac{\mu_i}{\mu_1},\ 1 \le i \le s; \quad \alpha_{k(1)} = \frac{\lambda_k}{\mu_1},\ h_1 + 1 \le k \le m + 1;$$

$$\tilde{p}_{k(ij)} = \frac{p_{k(ij)}}{p_0},\ 1 \le k \le h_2 - 1,\ 1 \le i \le s,\ i \le j \le s;$$

$$\tilde{q}_{k(ij)} = \frac{q_{k(ij)}}{p_0},\ h_1 + 1 \le k \le m + 2,\ 1 \le i \le s,\ i \le j \le s;$$

$$\tilde{p}_{k(is)} = p_{ki},\ 2 \le k \le h_2 - 1,\quad \tilde{q}_{k(is)} = q_{ki},\ h_1 + 1 \le k \le m + 2,\ 1 \le i \le s,$$

and using the system of algebraic equations, we find:

$$\tilde{p}_{1(0s)} = \alpha_s, \quad \tilde{p}_{1(01)} = \frac{1}{\alpha_1 + 1}(\alpha_1 + \eta_s p_{21});$$

$$\tilde{p}_{1(0j)} = \frac{1}{\alpha_1 + \eta_j}\left(\eta_{j-1}\tilde{p}_{1(0,j-1)} + \eta_s p_{2j}\right), \quad 2 \le j \le s - 1;$$

$$\tilde{p}_{k(11)} = \frac{1}{\alpha_1 + 2}\left(\alpha_1 \tilde{p}_{k-1(11)} + \eta_s p_{k+1,1}\right), \quad 2 \le k \le h_2 - 2,\ k \ne h_1;$$

$$\tilde{p}_{h_1(11)} = \frac{1}{\alpha_1 + 2}\left(\alpha_1 \tilde{p}_{h_1-1(11)} + \eta_s (p_{h_1+1,1} + q_{h_1+1,1})\right), \quad \tilde{p}_{h_2-1(11)} = \frac{\alpha_1 \tilde{p}_{h_2-2(11)}}{\alpha_1 + 2};$$

$$\tilde{q}_{h_1+1(11)} = \frac{\eta_s q_{h_1+2,1}}{\alpha_{h_1+1(1)}+2}, \quad \tilde{q}_{h_2(11)} = \frac{1}{\alpha_{h_2(1)}+2}\left(\alpha_{h_2-1(1)}\tilde{q}_{h_2-1(11)} + \alpha_1\tilde{p}_{h_2-1(11)} + \eta_s q_{h_2+1,1}\right);$$

$$\tilde{q}_{k(11)} = \frac{1}{\alpha_{k(1)}+2}\left(\alpha_{k-1(1)}\tilde{q}_{k-1(11)} + \eta_s q_{k+1,1}\right), \quad h_1+2 \le k \le m+1,\ k \ne h_2;$$

$$\tilde{q}_{m+2(11)} = \frac{\alpha_{m+1(1)}}{2}\tilde{q}_{m+1(11)};$$

$$\tilde{p}_{k(12)} = \frac{1}{\alpha_1+\eta_2+1}\left(\alpha_1\tilde{p}_{k-1(12)} + 2\tilde{p}_{k(11)} + \eta_s p_{k+1,2}\right), \quad 2 \le k \le h_2-2,\ k \ne h_1;$$

$$\tilde{p}_{h_1(12)} = \frac{1}{\alpha_1+\eta_2+1}\left(\alpha_1\tilde{p}_{h_1-1(12)} + 2\tilde{p}_{h_1(11)} + \eta_s(p_{h_1+1,2} + q_{h_1+1,2})\right);$$

$$\tilde{p}_{h_2-1(12)} = \frac{1}{\alpha_1+\eta_2+1}\left(\alpha_1\tilde{p}_{h_2-2(12)} + 2\tilde{p}_{h_2-1(11)}\right);$$

$$\tilde{q}_{h_1+1(12)} = \frac{1}{\alpha_{h_1+1(1)}+\eta_2+1}\left(2\tilde{q}_{h_1+1(11)} + \eta_s q_{h_1+2,2}\right), \quad \tilde{q}_{m+2(12)} = \frac{1}{\eta_2+1}\left(\alpha_{m+1(1)}\tilde{q}_{m+1(12)} + 2\tilde{q}_{m+2(11)}\right);$$

$$\tilde{q}_{k(12)} = \frac{1}{\alpha_{k(1)}+\eta_2+1}\left(\alpha_{k-1(1)}\tilde{q}_{k-1(12)} + 2\tilde{q}_{k(11)} + \eta_s q_{k+1,2}\right), \quad h_1+2 \le k \le m+1,\ k \ne h_2;$$

$$\tilde{q}_{h_2(12)} = \frac{1}{\alpha_{h_2(1)}+\eta_2+1}\left(\alpha_{h_2-1(1)}\tilde{q}_{h_2-1(12)} + \alpha_1\tilde{p}_{h_2-1(12)} + 2\tilde{q}_{h_2(11)} + \eta_s q_{h_2+1,2}\right);$$

$$\tilde{p}_{k(1j)} = \frac{1}{\alpha_1+\eta_j+1}\left(\alpha_1\tilde{p}_{k-1(1j)} + \eta_{j-1}\tilde{p}_{k(1,j-1)} + \eta_s p_{k+1,j}\right),$$
$$2 \le k \le h_2-1,\ k \ne h_1;\ 3 \le j \le s-1;$$

$$\tilde{p}_{h_1(1j)} = \frac{1}{\alpha_1+\eta_j+1}\left(\alpha_1\tilde{p}_{h_1-1(1j)} + \eta_{j-1}\tilde{p}_{h_1(1,j-1)} + \eta_s(p_{h_1+1,j} + q_{h_1+1,j})\right), \ 3 \le j \le s-1;$$

$$\tilde{p}_{h_2-1(1j)} = \frac{1}{\alpha_1+\eta_j+1}\left(\alpha_1\tilde{p}_{h_2-2(1j)} + \eta_{j-1}\tilde{p}_{h_2-1(1,j-1)}\right), \ 3 \le j \le s-1;$$

$$\tilde{q}_{h_1+1(1j)} = \frac{1}{\alpha_1+\eta_j+1}\left(\eta_{j-1}\tilde{q}_{h_1+1(1,j-1)} + \eta_s q_{h_1+2,j}\right), \ 3 \le j \le s-1;$$

$$\tilde{q}_{k(1j)} = \frac{1}{\alpha_{k(1)}+\eta_j+1}\left(\alpha_{k-1(1)}\tilde{q}_{k-1(1j)} + \eta_{j-1}\tilde{q}_{k(1,j-1)} + \eta_s q_{k+1,j}\right),$$
$$h_1+2 \le k \le m+1,\ k \ne h_2;\ 3 \le j \le s-1;$$

$$\tilde{q}_{h_2(1j)} = \frac{1}{\alpha_{h_2(1)}+\eta_j+1}\left(\alpha_{h_2-1(1)}\tilde{q}_{h_2-1(1j)} + \alpha_1\tilde{p}_{h_2-1(1j)} + \eta_{j-1}\tilde{q}_{h_2(1,j-1)} + \eta_s q_{h_2+1,j}\right), \ 3 \le j \le s-1;$$

$$\tilde{q}_{m+2(1j)} = \frac{1}{\eta_j+1}\left(\alpha_{m+1(1)}\tilde{q}_{m+1(1j)} + \eta_{j-1}\tilde{q}_{m+2(1,j-1)}\right), \ 3 \le j \le s-1;$$

$$\tilde{p}_{2(ii)} = \frac{\eta_{i-1}}{\alpha_1 + 2\eta_i}\tilde{p}_{2(i-1,i)}, \quad 2 \le i \le s-1;$$

$$\tilde{p}_{k(ii)} = \frac{1}{\alpha_1 + 2\eta_i}\left(\alpha_1 \tilde{p}_{k-1(ii)} + \eta_{i-1}\tilde{p}_{k(i-1,i)}\right), \ 3 \le k \le h_2 - 1, \ 2 \le i \le s-1;$$

$$\tilde{q}_{h_1+1(ii)} = \frac{\eta_{i-1}\tilde{q}_{h_1+1(i-1,i)}}{\alpha_{h_1+1(1)} + 2\eta_i}, \quad 2 \le i \le s-1;$$

$$\tilde{q}_{k(ii)} = \frac{1}{\alpha_{k(1)} + 2\eta_i}\left(\alpha_{k-1(1)}\tilde{q}_{k-1(ii)} + \eta_{i-1}\tilde{q}_{k(i-1,i)}\right), \ h_1 + 1 \le k \le m+1, \ k \ne h_2; \ 2 \le i \le s-1;$$

$$\tilde{q}_{h_2(ii)} = \frac{1}{\alpha_{h_2(1)} + 2\eta_i}\left(\alpha_{h_2-1(1)}\tilde{q}_{h_2-1(ii)} + \alpha_1 \tilde{p}_{h_2-1(ii)} + \eta_{i-1}\tilde{q}_{h_2(i-1,i)}\right), \ 2 \le i \le s-1;$$

$$\tilde{q}_{m+2(ii)} = \frac{1}{2\eta_i}\left(\alpha_{m+1(1)}\tilde{q}_{m+1(ii)} + \eta_{i-1}\tilde{q}_{m+2(i-1,i)}\right), \ 2 \le i \le s-1;$$

$$\tilde{p}_{2(i,i+1)} = \frac{1}{\alpha_1 + \eta_i + \eta_{i+1}}\left(2\eta_i \tilde{p}_{2(ii)} + \eta_{i-1}\tilde{p}_{2(i-1,i+1)}\right), \ 2 \le i \le s-2;$$

$$\tilde{p}_{2(ij)} = \frac{1}{\alpha_1 + \eta_i + \eta_j}\left(\eta_{i-1}\tilde{p}_{2(i-1,j)} + \eta_{j-1}\tilde{p}_{2(i,j-1)}\right), \ 2 \le i \le s-3, \ i+2 \le j \le s-1;$$

$$\tilde{p}_{k(i,i+1)} = \frac{1}{\alpha_1 + \eta_i + \eta_{i+1}}\left(\alpha_1 \tilde{p}_{k-1(i,i+1)} + 2\eta_i \tilde{p}_{k(ii)} + \eta_{i-1}\tilde{p}_{k(i-1,i+1)}\right),$$
$$3 \le k \le h_2 - 1, \ 2 \le j \le s-2;$$

$$\tilde{q}_{h_1+1(i,i+1)} = \frac{1}{\alpha_{h_1+1(1)} + \eta_i + \eta_{i+1}}\left(2\eta_i \tilde{q}_{h_1+1(ii)} + \eta_{i-1}\tilde{q}_{h_1+1(i-1,i+1)}\right), \ 2 \le j \le s-2;$$

$$\tilde{q}_{k(i,i+1)} = \frac{1}{\alpha_{k(1)} + \eta_i + \eta_{i+1}}\left(\alpha_{k-1(1)}\tilde{q}_{k-1(i,i+1)} + 2\eta_i \tilde{q}_{k(ii)} + \eta_{i-1}\tilde{q}_{k(i-1,i+1)}\right),$$
$$h_1 + 2 \le k \le m+1, \ k \ne h_2; \ 2 \le j \le s-2;$$

$$\tilde{q}_{h_2(i,i+1)} = \frac{1}{\alpha_{h_2(1)} + \eta_i + \eta_{i+1}}\left(\alpha_{h_2-1(1)}\tilde{q}_{h_2-1(i,i+1)} + \alpha_1 \tilde{p}_{h_2-1(i,i+1)} + 2\eta_i \tilde{q}_{h_2(ii)} + \eta_{i-1}\tilde{q}_{h_2(i-1,i+1)}\right),$$
$$2 \le i \le s-2;$$

$$\tilde{q}_{m+2(i,i+1)} = \frac{1}{\eta_i + \eta_{i+1}}\left(\alpha_{m+1(1)}\tilde{q}_{m+1(i,i+1)} + 2\eta_i \tilde{q}_{m+2(ii)} + \eta_{i-1}\tilde{q}_{m+2(i-1,i+1)}\right), \ 2 \le i \le s-2;$$

$$\tilde{p}_{k(ij)} = \frac{1}{\alpha_1 + \eta_i + \eta_j}\left(\alpha_1 \tilde{p}_{k-1(ij)} + \eta_{i-1}\tilde{p}_{k(i-1,j)} + \eta_{j-1}\tilde{p}_{k(i,j-1)}\right),$$
$$3 \le k \le h_2 - 1, \ 2 \le i \le s-3, \ i+2 \le j \le s-1;$$

$$\tilde{q}_{h_1+1(ij)} = \frac{1}{\alpha_{h_1+1(1)} + \eta_i + \eta_j}\left(\eta_{i-1}\tilde{q}_{h_1+1(i-1,j)} + \eta_{j-1}\tilde{q}_{h_1+1(i,j-1)}\right), \ 2 \le i \le s-3, \ i+2 \le j \le s-1;$$

$$\tilde{q}_{k(ij)} = \frac{1}{\alpha_{k(1)} + \eta_i + \eta_j} \left(\alpha_{k-1(1)} \tilde{q}_{k-1(ij)} + \eta_{i-1} \tilde{q}_{k(i-1,j)} + \eta_{j-1} \tilde{q}_{k(i,j-1)} \right),$$
$$h_1 + 2 \le k \le m+1,\ k \ne h_2;\ 2 \le i \le s-3,\ i+2 \le j \le s-1;$$
$$\tilde{q}_{h_2(ij)} = \frac{1}{\alpha_{h_2(1)} + \eta_i + \eta_j} \left(\alpha_{h_2-1(1)} \tilde{q}_{h_2-1(ij)} + \alpha_1 \tilde{p}_{h_2-1(ij)} + \eta_{i-1} \tilde{q}_{h_2(i-1,j)} + \eta_{j-1} \tilde{q}_{h_2(i,j-1)} \right), \quad (2)$$
$$2 \le i \le s-3,\ i+2 \le j \le s-1;$$
$$\tilde{q}_{m+2(ij)} = \frac{1}{\eta_i + \eta_j} \left(\alpha_{m+1(1)} \tilde{q}_{m+1(ij)} + \eta_{i-1} \tilde{q}_{m+2(i-1,j)} + \eta_{j-1} \tilde{q}_{m+2(i,j-1)} \right),$$
$$2 \le i \le s-3,\ i+2 \le j \le s-1.$$

Recurrence relations (2) allow one to compute $\tilde{p}_{k(ij)}$ and $\tilde{q}_{k(ij)}$ as linear functions of the unknown parameters $p_{ki}\,(2 \le k \le h_2 - 1,\ 1 \le i \le s)$ and $q_{ki}\ (h_1 + 1 \le k \le m+2,\ 1 \le i \le s)$ in the following order:

$\tilde{p}_{1(0s)}$; $\tilde{p}_{1(0j)}$, $1 \le j \le s-1$; $\tilde{p}_{k(11)}$, $2 \le k \le h_2 - 1$; $\tilde{q}_{k(11)}$, $h_1 + 1 \le k \le m+2$;
$\tilde{p}_{k(12)}$, $2 \le k \le h_2 - 1$; $\tilde{q}_{k(12)}$, $h_1 + 1 \le k \le m+2$;
$\tilde{p}_{k(13)}, \tilde{p}_{k(14)}, \dots, \tilde{p}_{k(1,s-1)}$, $2 \le k \le h_2 - 1$; $\tilde{q}_{k(13)}, \tilde{q}_{k(14)}, \dots, \tilde{q}_{k(1,s-1)}$, $h_1 + 1 \le k \le m+1$;
$\tilde{q}_{m+2(1j)}$, $3 \le j \le s-1$;
$\tilde{p}_{k(22)}$, $2 \le k \le h_2 - 1$; $\tilde{q}_{k(22)}$, $h_1 + 1 \le k \le m+2$; $\tilde{p}_{2(2j)}$, $3 \le j \le s-1$;
$\tilde{p}_{k(23)}$, $3 \le k \le h_2 - 1$; $\tilde{q}_{k(23)}$, $h_1 + 1 \le k \le m+2$;
$\tilde{p}_{k(24)}, \tilde{p}_{k(25)}, \dots, \tilde{p}_{k(2,s-1)}$, $3 \le k \le h_2 - 1$; $\tilde{q}_{k(24)}, \tilde{q}_{k(25)}, \dots, \tilde{q}_{k(2,s-1)}$, $h_1 + 1 \le k \le m+1$;
$\tilde{q}_{m+2(2j)}$, $4 \le j \le s-1$;
$\tilde{p}_{k(33)}$, $2 \le k \le h_2 - 1$; $\tilde{q}_{k(33)}$, $h_1 + 1 \le k \le m+2$; $\tilde{p}_{2(3j)}$, $4 \le j \le s-1$;
$\tilde{p}_{k(34)}$, $3 \le k \le h_2 - 1$; $\tilde{q}_{k(34)}$, $h_1 + 1 \le k \le m+2$;
$\tilde{p}_{k(35)}, \tilde{p}_{k(36)}, \dots, \tilde{p}_{k(3,s-1)}$, $3 \le k \le h_2 - 1$; $\tilde{q}_{k(35)}, \tilde{q}_{k(36)}, \dots, \tilde{q}_{k(3,s-1)}$, $h_1 + 1 \le k \le m+1$;
$\tilde{q}_{m+2(3j)}$, $5 \le j \le s-1$;
....
$\tilde{p}_{k(ii)}$, $2 \le k \le h_2 - 1$; $\tilde{q}_{k(ii)}$, $h_1 + 1 \le k \le m+2$; $\tilde{p}_{2(ij)}$, $i+1 \le j \le s-1$;
$\tilde{p}_{k(i,i+1)}$, $3 \le k \le h_2 - 1$; $\tilde{q}_{k(i,i+1)}$, $h_1 + 1 \le k \le m+2$;
$\tilde{p}_{k(i,i+2)}, \tilde{p}_{k(i,i+3)}, \dots, \tilde{p}_{k(i,s-1)}$, $3 \le k \le h_2 - 1$; $\tilde{q}_{k(i,i+2)}, \tilde{q}_{k(i,i+3)}, \dots, \tilde{q}_{k(i,s-1)}$,
$h_1 + 1 \le k \le m+1$;
$\tilde{q}_{m+2(ij)}$, $i+2 \le j \le s-1$;
....

$$\tilde{p}_{k(s-2,s-2)},\ 2 \le k \le h_2 - 1; \quad \tilde{q}_{k(s-2,s-2)},\ h_1 + 1 \le k \le m + 2;$$
$$\tilde{p}_{k(s-2,s-1)},\ 3 \le k \le h_2 - 1; \quad \tilde{q}_{k(s-2,s-1)},\ h_1 + 1 \le k \le m + 2;$$
$$\tilde{p}_{k(s-1,s-1)},\ 2 \le k \le h_2 - 1; \quad \tilde{q}_{k(s-1,s-1)},\ h_1 + 1 \le k \le m + 2.$$

To determine unknown parameters p_{ki} $(2 \le k \le h_2 - 1,\ 1 \le i \le s)$ and q_{ki} $(h_1 + 1 \le k \le m + 2,\ \ 1 \le i \le s)$, we use the system that consists of $s(h_2 - h_1 + m)$ equations that have not been involved in obtaining recurrence relations (2),

$$-(\lambda + \mu_1 + \mu_s) p_{k(1s)} + \lambda p_{k-1(1s)} + 2\mu_s p_{k+1(ss)} + \mu_{s-1} p_{k(1,s-1)} = 0,\ \ 2 \le k \le h_2 - 2,\ k \ne h_1;$$
$$-(\lambda + \mu_1 + \mu_s) p_{h_1(1s)} + \lambda p_{h_1-1(1s)} + 2\mu_s (p_{h_1+1(ss)} + q_{h_1+1(ss)}) + \mu_{s-1} p_{h_1(1,s-1)} = 0;$$
$$-(\lambda + \mu_1 + \mu_s) p_{h_2-1(1s)} + \lambda p_{h_2-2(1s)} + \mu_{s-1} p_{h_2-1(1,s-1)} = 0;$$
$$-(\lambda_{h_1+1} + \mu_1 + \mu_s) q_{h_1+1(1s)} + 2\mu_s q_{h_1+2(ss)} + \mu_{s-1} q_{h_1+1(1,s-1)} = 0;$$
$$-(\lambda_k + \mu_1 + \mu_s) q_{k(1s)} + \lambda_{k-1} q_{k-1(1s)} + 2\mu_s q_{k+1(ss)} + \mu_{s-1} q_{k(1,s-1)} = 0,$$
$$h_1 + 2 \le k \le m + 1,\ k \ne h_2;$$
$$-(\lambda_{h_2} + \mu_1 + \mu_s) q_{h_2(1s)} + \lambda_{h_2-1} q_{h_2-1(1s)} + \lambda p_{h_2-1(1s)} + 2\mu_s q_{h_2+1(ss)} + \mu_{s-1} q_{h_2(1,s-1)} = 0;$$
$$-(\mu_1 + \mu_s) q_{m+2(1s)} + \lambda_{m+1} q_{m+1(1s)} + \mu_{s-1} q_{m+2(1,s-1)} = 0;$$
$$-(\lambda + 2\mu_s) p_{2(ss)} + \mu_{s-1} p_{2(s-1,s)} = 0;$$
$$-(\lambda + \mu_{s-1} + \mu_s) p_{2(s-1,s)} + 2\mu_{s-1} p_{2(s-1,s-1)} + \mu_{s-2} p_{2(s-2,s)} = 0;$$
$$-(\lambda + \mu_i + \mu_s) p_{2(is)} + \mu_{i-1} p_{2(i-1,s)} + \mu_{s-1} p_{2(i,s-1)} = 0, \quad 2 \le i \le s-2;$$
$$-(\lambda + 2\mu_s) p_{k(ss)} + \lambda p_{k-1(ss)} + \mu_{s-1} p_{k(s-1,s)} = 0, \quad 3 \le k \le h_2 - 1;$$
$$-(\lambda_{h_1+1} + 2\mu_s) q_{h_1+1(ss)} + \mu_{s-1} q_{h_1+1(s-1,s)} = 0;$$
$$-(\lambda_k + 2\mu_s) q_{k(ss)} + \lambda_{k-1} q_{k-1(ss)} + \mu_{s-1} q_{k(s-1,s)} = 0, \quad h_1 + 2 \le k \le m + 1,\ k \ne h_2;$$
$$-(\lambda_{h_2} + 2\mu_s) q_{h_2(ss)} + \lambda_{h_2-1} q_{h_2-1(ss)} + \lambda p_{h_2-1(ss)} + \mu_{s-1} q_{h_2(s-1,s)} = 0;$$
$$-2\mu_s q_{m+2(ss)} + \lambda_{m+1} q_{m+1(ss)} + \mu_{s-1} q_{m+2(s-1,s)} = 0;$$
$$-(\lambda + \mu_{s-1} + \mu_s) p_{k(s-1,s)} + \lambda p_{k-1(s-1,s)} + 2\mu_{s-1} p_{k(s-1,s-1)} + \mu_{s-2} p_{k(s-2,s)} = 0,$$
$$3 \le k \le h_2 - 1;$$
$$-(\lambda_{h_1+1} + \mu_{s-1} + \mu_s) q_{h_1+1(s-1,s)} + 2\mu_{s-1} q_{h_1+1(s-1,s-1)} + \mu_{s-2} q_{h_1+1(s-2,s)} = 0;$$
$$-(\lambda_k + \mu_{s-1} + \mu_s) q_{k(s-1,s)} + \lambda_{k-1} q_{k-1(s-1,s)} + 2\mu_{s-1} q_{k(s-1,s-1)} + \mu_{s-2} q_{k(s-2,s)} = 0,$$
$$h_1 + 2 \le k \le m + 1,\ k \ne h_2;$$
$$-(\lambda_{h_2} + \mu_{s-1} + \mu_s) q_{h_2(s-1,s)} + \lambda_{h_2-1} q_{h_2-1(s-1,s)} + \lambda p_{h_2-1(s-1,s)} +$$
$$+ 2\mu_{s-1} q_{h_2(s-1,s-1)} + \mu_{s-2} q_{h_2(s-2,s)} = 0;$$
$$-(\mu_{s-1} + \mu_s) q_{m+2(s-1,s)} + \lambda_{m+1} q_{m+1(s-1,s)} + 2\mu_{s-1} q_{m+2(s-1,s-1)} + \mu_{s-2} q_{m+2(s-2,s)} = 0;$$

$$-(\lambda+\mu_i+\mu_s)p_{k(is)}+\lambda p_{k-1(is)}+\mu_{i-1}p_{k(i-1,s)}+\mu_{s-1}p_{k(i,s-1)}=0,$$
$$3\le k\le h_2-1;\ 2\le i\le s-2;$$
$$-(\lambda_{h_1+1}+\mu_i+\mu_s)q_{h_1+1(is)}+\mu_{i-1}q_{h_1+1(i-1,s)}+\mu_{s-1}q_{h_1+1(i,s-1)}=0,\ 2\le i\le s-2;$$
$$-(\lambda_k+\mu_i+\mu_s)q_{k(is)}+\lambda_{k-1}q_{k-1(is)}+\mu_{i-1}q_{k(i-1,s)}+\mu_{j-1}q_{k(i,s-1)}=0,$$
$$h_1+2\le k\le m+1,\ k\ne h_2;\ 2\le i\le s-2;$$
$$-(\lambda_{h_2}+\mu_i+\mu_s)q_{h_2(is)}+\lambda_{h_2-1}q_{h_2-1(is)}+\lambda p_{h_2-1(is)}+\mu_{i-1}q_{h_2(i-1,s)}+\mu_{j-1}q_{h_2(i,s-1)}=0,$$
$$2\le i\le s-2;$$
$$-(\mu_i+\mu_s)q_{m+2(is)}+\lambda_{m+1}q_{m+1(is)}+\mu_{i-1}q_{m+2(i-1,s)}+\mu_{s-1}q_{m+2(i,s-1)}=0,\ 2\le i\le s-2.$$

We use normalization condition (1) and determine the steady-state probabilities by the formulas

$$p_0=\left(1+\sum_{j=1}^{s}\tilde p_{1(0j)}+\sum_{k=2}^{h_2-1}\sum_{i=1}^{s}\sum_{j=i}^{s}\tilde p_{k(ij)}+\sum_{k=h_1+1}^{m+2}\sum_{i=1}^{s}\sum_{j=i}^{s}\tilde q_{k(ij)}\right)^{-1},$$

$$p_k=p_0\tilde p_k,\quad 1\le k\le m+2;\quad \tilde p_1=\sum_{j=1}^{s}\tilde p_{1(0j)};\quad \tilde p_k=\sum_{i=1}^{s}\sum_{j=i}^{s}\tilde p_{k(ij)},\ 2\le k\le h_1;$$

$$\tilde p_k=\sum_{i=1}^{s}\sum_{j=i}^{s}(\tilde p_{k(ij)}+\tilde q_{k(ij)}),\ h_1+1\le k\le h_2-1;\quad \tilde p_k=\sum_{i=1}^{s}\sum_{j=i}^{s}\tilde q_{k(ij)},\ h_2\le k\le m+2.$$

Here p_k is steady-state probability of presence k customers in the system.

We calculate the steady-state characteristics - the average number of customers in the system $\mathbf{E}(C)$, the average queue length $\mathbf{E}(Q)$, average waiting time $\mathbf{E}(W)$ and service probability $\mathbf{P}_{\text{sv}}$ - by the formulas

$$\mathbf{E}(C)=\sum_{k=1}^{m+2}kp_k,\quad \mathbf{E}(Q)=\sum_{k=3}^{m+2}(k-2)p_k,\quad \mathbf{E}(W)=\frac{\mathbf{E}(Q)}{\lambda\mathbf{P}_{\text{sv}}},$$
$$\mathbf{P}_{\text{sv}}=\frac{\bar\mu\big(2(1-p_0)-p_1\big)}{\lambda},\quad \bar\mu=\left(\sum_{i=1}^{s}\frac{1}{\mu_i}\right)^{-1}.$$

3. The system without restrictions on the queue length

For the $M(h_1,h_2)/E_s/2/\infty$ system, any constraint on the queue length is absent and, for the existence of stationary distribution of the number of customers in the system, the condition $\lambda\tilde\beta<2\bar\mu$ must be satisfied. Determining approximate values of steady-state probabilities p_k is reduced to the use of recurrence relations (2) for large values of *m*. We choose the number $N = m + 2$ so large that one of the conditions (or each of these conditions) specifying the accuracy of determining steady-state probabilities is fulfilled. These conditions can be specified, for example, in the form

$$\mathbf{E}(C)_{(N)} - \mathbf{E}(C)_{(N-1)} < \varepsilon_1, \qquad \mathbf{E}(Q)_{(N)} - \mathbf{E}(Q)_{(N-1)} < \varepsilon_2. \tag{3}$$

Here ε_1 and ε_2 are positive numbers specifying the required accuracy of computations; $\mathbf{E}(C)_{(N)}$ and $\mathbf{E}(Q)_{(N)}$ are approximate values of steady-state characteristics $\mathbf{E}(C)$ and $\mathbf{E}(Q)$, computed using steady-state probabilities $p_{k(N)}$ $(0 \le k \le N)$; $p_{k(N)}$ is an approximate value of a steady-state probability p_k, which is obtained as a result of truncation of an infinite system of equations for steady-state probabilities.

4. Numerical examples

Let us consider examples of determining steady-state characteristics of the $M(h_1,h_2)/E_5/2/\infty$ queuing system for different values of the thresholds h_1 and h_2: 1) $h_1 = 6, h_2 = 8$; 2) $h_1 = 6, h_2 = 10$; 3) $h_1 = 6, h_2 = 12$. Let $\lambda = 1$; $\mu_i = 2.5$, $1 \le i \le 5$; the probabilities β_k for the dropping mode be set according to the rule: $\beta_k = \beta = 0.8$ for $k \ge h_1 + 1$. For comparison, we calculate the stationary characteristics of the $M/E_5/2/43$ system, which does not apply the random dropping of customers.

The values of the steady-state probabilities p_k and stationary characteristics of the $M(h_1,h_2)/E_5/2/\infty$ system for cases 1-3, found using the recurrence relations obtained in this paper, as well as of the $M/E_5/2/43$ system, are presented in Tables 1 and 2. In order to verify the obtained values, Table 2 contains the computing results evaluated by the GPSS World simulation system [14] for the simulation time value $t = 10^7$.

In computing approximate values of steady-state probabilities p_k, the value of N was selected so large that conditions (3) were satisfied when $\varepsilon = 10^{-5}$. The obtained minimal values of N for cases 1-3 are equal to 45, 45 and 47 respectively.

Analyzing the results, presented in Table 2, we see that the control of the input flow rate with the help of random dropping of customers makes it possible to considerably reduce the average queue length with an insignificant decrease in the system throughput. Thus, the decrease in the average queue length in the $M(h_1,h_2)/E_5/2/\infty$ system (the case of $h_1 = 6, h_2 = 8$) in comparison with the $M/E_5/2/43$ system amounts to 81.6%, with a decrease in the relative throughput by 5.3%. If the value of the threshold h_2 increases, leaving the same value of h_1, then $\mathbf{E}(Q)$ and $\mathbf{P}_{\text{sv}}$ increase.

Table 1

Stationary distribution of the number of customers in the systems

k	Values of the steady-state probabilities p_k			
	$M/E_5/2/43$	$M(h_1,h_2)/E_5/2/\infty$, $h_1=6$, $h_2=8$	$M(h_1,h_2)/E_5/2/\infty$, $h_1=6$, $h_2=10$	$M(h_1,h_2)/E_5/2/\infty$, $h_1=6$, $h_2=12$
0	0.006270	0.030479	0.027564	0.025071
1	0.014577	0.070856	0.064078	0.058284
2	0.019569	0.095119	0.086020	0.078242
3	0.021658	0.105274	0.095203	0.086595
4	0.022351	0.108637	0.098247	0.089366
5	0.022544	0.109648	0.099131	0.090157
6	0.022589	0.107500	0.097400	0.088673
7	0.022597	0.101170	0.096390	0.088693
8	0.022598	0.080393	0.087893	0.083804
9	0.022598	0.058035	0.072205	0.075011
10	0.022598	0.040450	0.052594	0.063798
11	0.022598	0.028147	0.037520	0.050642
12	0.022598	0.019577	0.026111	0.036400
13	0.022598	0.013616	0.018162	0.025953
14	0.022598	0.009470	0.012632	0.018059
15	0.022598	0.006586	0.008785	0.012561
20	0.022598	0.001072	0.001430	0.002044
30	0.022598	0.000028	0.000038	0.000054
40	0.022598	$7.517\cdot10^{-7}$	$1.003\cdot10^{-6}$	$1.434\cdot10^{-6}$
45	0.013559	$8.236\cdot10^{-8}$	$1.099\cdot10^{-7}$	$2.354\cdot10^{-7}$

Table 2

Stationary characteristics of the systems

System	Method	Values of the stationary characteristics			
		$\mathbf{E}(C)$	$\mathbf{E}(Q)$	$\mathbf{E}(W)$	$\mathbf{P}_{sv}$
$M/E_5/2/43$	Recurrence	22.88257	20.90968	21.19709	0.98644
	GPSS World	22.896	20.923	21.212	0.986
$M(h_1.h_2)/E_5/2/\infty$, $h_1=6$, $h_2=8$	Recurrence	5.71082	3.84263	4.11376	0.93409
	GPSS World	5.717	3.848	4.118	0.935
$M(h_1,h_2)/E_5/2/\infty$, $h_1=6$, $h_2=10$	Recurrence	6.18609	4.30530	4.57817	0.94040
	GPSS World	6.179	4.299	4.571	0.941
$M(h_1,h_2)/E_5/2/\infty$, $h_1=6$, $h_2=12$	Recurrence	6.71627	4.82469	5.10125	0.94579
	GPSS World	6.712	4.821	5.097	0.946

5. Conclusions

Using the method of fictitious phases, an algorithm for calculating the stationary distribution of the number of customers in the $M/E_s/2/m$ systems with hysteretic strategy of the random dropping of customers, inclusive of the case $m = \infty$, is constructed. The obtained recurrence relations are used for the direct computation of the solutions of the algebraic system for the steady-state probabilities, which makes it possible to reduce the amount of computations in comparison with the direct or iterative classical methods. Using the obtained recurrence relations makes it possible to reduce the number of solved equations from $(s+1)\left(2+s(h_2-h_1+m)\right)/2$ to $s(h_2-h_1+m)$.

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