JAM
CM
Journal of Applied Mathematics and Computational Mechanics 2019, 18(1), 69-76
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2019.1.06 e-ISSN 2353-0588

EDGE PRODUCT CORDIAL LABELING OF SOME GRAPHS

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Received: 12 October 2018; Accepted: 20 May 2019

Abstract. For a graph $G = (V(G),E(G))$ having no isolated vertex, a function $f : E(G) \to \{0,1\}$ is called an edge product cordial labeling of graph G, if the induced vertex labeling function defined by the product of labels of incident edges to each vertex be such that the number of edges with label 0 and the number of edges with label 1 differ by at the most 1 and the number of vertices with label 0 and the number of vertices with label 1 also differ by at the most 1. In this paper we discuss the edge product cordial labeling of the graphs $W_n^{(t)}$, PS_n and DPS_n.

MSC 2010: *05C78,05C76*
Keywords: *graph labeling, edge product cordial labeling*

1. Introduction

We begin with a simple, finite, undirected graph $G = (V(G),E(G))$ having no isolated vertex where $V(G)$ and $E(G)$ denote the vertex set and the edge set respectively, $|V(G)|$ and $|E(G)|$ denote the number of vertices and edges respectively. For all other terminology we follow Gross [1]. We will give a brief summary of definitions which are useful for the present work.

Definition 1 A *graph labeling* is an assignment of integers to the vertices or edges or both(edges and vertices) subject to the certain conditions. If the domain of the mapping is the set of vertices(or edges) then the labeling is called vertex (or edge) labeling.
For an extensive survey on graph labeling and bibliography references, we refer to Gallian [2]. □

Definition 2 For a graph G, the edge labeling function is defined as $f : E(G) \to \{0,1\}$ and induced vertex labeling function $f^* : V(G) \to \{0,1\}$ is given as if $e_1, e_2, \ldots,$ e_k are all the edges incident to the vertex v then $f^*(v) = f(e_1)f(e_2)\ldots f(e_k)$.
Let $v_f(i)$ be the number of vertices of G having label i under f^* and $e_f(i)$ be the number of edges of G having label i under f for $i = 0,1$.

f is called an *edge product cordial labeling* of graph G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called an *edge product cordial* if it admits an edge product cordial labeling. □

In 1987 Cahit, [3], first established cordial labeling. In 2004, Sundaram et al. [4] introduced product cordial labeling. In 2012, Vaidya and Barasara [5] introduced the concept of edge product cordial labeling as an edge analogue of the product cordial labeling in which they have investigated that the following graphs are edge product cordial: C_n for n odd; trees with order greater than 2; unicyclic graphs of odd order; crown $C_n \bigodot K_1$; armed crowns $C_m \bigodot P_n$; helms; closed helms; webs; flowers; gears G_n and shells S_n for odd n. They also proved that the following graphs are not edge product cordial: C_n for n even; wheels; shells S_n for even n.
Vaidya and Barasara [6] discussed edge product cordial labeling for some snake related graphs. In [7], Vaidya and Barasara discussed product and edge product cordial labelings of the degree splitting graphs of paths, shells, bistars, and gear graphs.
Prajapati and Patel [8] discussed edge product cordial labeling of some cycle related graphs.

Definition 3 The *wheel* W_n $(n \geq 3)$ is the graph obtained by adding a new vertex joining each of the vertices of C_n. The new vertex is called the apex vertex and the vertices corresponding to C_n are called rim vertices of W_n. The edges joining the rim vertices are called rim edges. □

Definition 4 The graph $W_n^{(t)}$ is a *one point union of t copies of* W_n with a rim vertex in common. □

Definition 5 The *Pentagonal Snake PS_n* is obtained from the path P_n by replacing every edge of a path by a cycle C_5. □

Definition 6 The *double Pentagonal Snake DPS_n* consists of two pentangonal snakes that have a common path. □

2. Main result

Theorem 1 *For $n \geq 3$, the $W_n^{(t)}$ is an edge product cordial if and only if t is even.* □

Proof: Let $e_{k,1}, e_{k,2}, \ldots, e_{k,n}$ be the consecutive rim edges of the k^{th} copy of W_n and $e'_{k,1}, e'_{k,2}, \ldots, e'_{k,n}$ be the consecutive spoke edges of the k^{th} copy of W_n. Let v be the common vertex of $W_n^{(t)}$. Let $v_{k,0}$ be the apex vertex of the k^{th} copy of wheel W_n and $v_{k,1}, v_{k,2}, \ldots, v_{k,n-1}$ be the remaining consecutive rim vertices of k^{th} copy of W_n. The edges $e_{k,1}, e_{k,n}$ and $e'_{k,1}$ of k^{th} copy of W_n are incident to v. Thus $\left|V(W_n^{(t)})\right| = tn + 1$ and $\left|E(W_n^{(t)})\right| = 2tn$. We consider the following two cases:

case 1: If t is even, define the mapping $f: E\left(W_n^{(t)}\right) \to \{0,1\}$ as follows:

$$f(e)=\begin{cases}1 & \text{if} \quad e=e_{i,j} \text{ for } i=1,2,\ldots,\frac{t}{2} \text{ and } j=1,2,\ldots,n;\\ 0 & \text{if} \quad e=e_{i,j} \text{ for } i=\frac{t}{2}+1,\frac{t}{2}+2,\ldots,t \text{ and } j=1,2,\ldots,n;\\ 1 & \text{if} \quad e=e'_{i,j} \text{ for } i=1,2,\ldots,\frac{t}{2} \text{ and } j=1,2,\ldots,n;\\ 0 & \text{if} \quad e=e'_{i,j} \text{ for } i=\frac{t}{2}+1,\frac{t}{2}+2,\ldots,t \text{ and } j=1,2,\ldots,n.\end{cases}$$

The induced vertex labeling function $f^*: V\left(W_n^{(t)}\right) \to \{0,1\}$ is given by:

$$f^*(v)=\prod_{k=1}^{t} f(e_{k,1}).f(e_{k,n}).f(e'_{k,1}).$$

$$f^*(v_{i,0})=\prod_{m=1}^{n} f(e'_{i,m}) \text{ for } \quad i=1,2,\ldots,t.$$

$$f^*(v_{i,j})=f(e_{i,j})f(e_{i,j+1})f(e'_{i,j+1}) \text{ for } \quad j=1,2,\ldots,n-1 \text{ and } i=1,2,\ldots,t.$$

From the above defined labeling pattern, we have
$v_f(0)=\left|\left\{v, v_{k,0}, v_{k,1},\ldots,v_{k,n-1} \text{ for } \frac{t}{2}+1\le k\le t\right\}\right|$ and
$v_f(1)=\left|\left\{v_{k,0}, v_{k,1},\ldots,v_{k,n-1} \text{ for } 1\le k\le \frac{t}{2}\right\}\right|$. So $v_f(0)=v_f(1)+1=\frac{tn}{2}+1$ and $e_f(0)=e_f(1)=tn$.
Thus $\left|v_f(0)-v_f(1)\right|\le 1$ and $\left|e_f(0)-e_f(1)\right|\le 1$. Thus f admits edge product cordial labeling on $W_n^{(t)}$. So $W_n^{(t)}$ is an edge product cordial graph for any n and t even.

case 2: If t is odd, then there are again two cases that arise:

[a] If n is odd and t is odd, then in order to satisfy the edge condition for the edge product cordial graph, it is essential to assign label 0 to tn edges out of $2tn$ edges. So in this context, the edges with label 0 will give rise to at least $\frac{tn+3}{2}$ vertices with label 0 and at most $\frac{tn-1}{2}$ vertices with label 1 out of $tn+1$ vertices. Therefore $|v_f(0)-v_f(1)|\ge 2$.

[b] If n is even and t is odd, then in order to satisfy the edge condition for the edge product cordial graph, it is essential to assign label 0 to tn edges out of $2tn$ edges. So in this context, the edges with label 0 will give rise to at least $\frac{tn+4}{2}$ vertices with label 0 and at most $\frac{tn-2}{2}$ vertices with label 1 out of $tn+1$ vertices. Therefore $|v_f(0)-v_f(1)|\ge 3$.

So, $W_n^{(t)}$ is not an edge product cordial graph for any n and t odd.

Example 1 An edge product cordial labeling of $W_4^{(4)}$ is shown in the following Figure 1. □

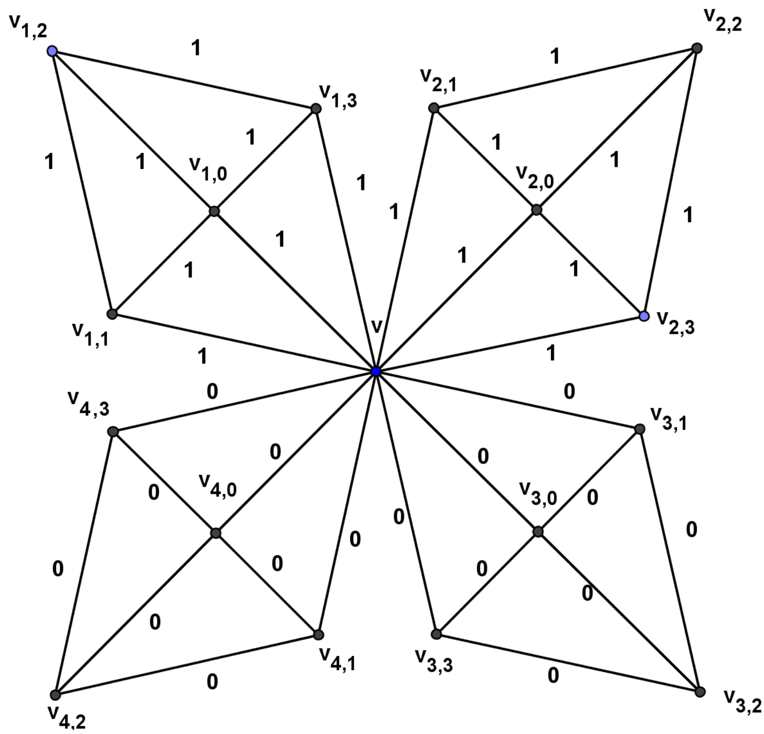

Fig. 1. Edge product cordial labeling of $W_4^{(4)}$

Theorem 2 *The graph PS_n is an edge product cordial graph.* □

Proof: Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of the path P_n. To construct PS_n from the path P_n with $e_i = v_i v_{i+1}$, for $i = 1, 2, \ldots, n-1$ join v_i to $w_i^{'}$ by the edge $e_{2i-1}^{'} = v_i w_i^{'}$ and v_{i+1} to $w_i^{''}$ by the edge $e_{2i}^{'} = v_{i+1} w_i^{''}$ for $i = 1, 2, \ldots, n-1$. Now join $w_i^{'}$ and $w_i^{''}$ to a single vertex w_i by the edge $e_{2i-1}^{''} = w_i^{'} w_i$ and $e_{2i}^{''} = w_i^{''} w_i$ for $i = 1, 2, \ldots, n-1$. Thus $|V(PS_n)| = 4n-3$ and $|E(PS_n)| = 5n-5$. Define the mapping $f : E(PS_n) \rightarrow \{0, 1\}$ as follows:

$$f(e) = \begin{cases} 1 & \text{if} \quad e = e_i \text{ for } i = 1, 2, \ldots, \left[\frac{n}{2}\right]; \\ 0 & \text{if} \quad e = e_i \text{ for } i = \left[\frac{n}{2}\right]+1, \left[\frac{n}{2}\right]+2, \ldots, n-1; \\ 1 & \text{if} \quad e = e_i^{'} \text{ for } i = 1, 2, \ldots, n-1; \\ 0 & \text{if} \quad e = e_i^{'} \text{ for } i = n, n+1, \ldots, 2n-2; \\ 1 & \text{if} \quad e = e_i^{''} \text{ for } i = 1, 2, \ldots, n-1; \\ 0 & \text{if} \quad e = e_i^{''} \text{ for } i = n, n+1, \ldots, 2n-2. \end{cases}$$

The induced vertex labeling function $f^*: V(PS_n) \to \{0,1\}$ is given by:

$$f^*(v_1) = f(e_1^{'})f(e_1).$$
$$f^*(v_n) = f(e_{2(n-1)}^{'})f(e_{n-1}).$$
$$f^*(v_i) = f(e_{i-1})f(e_i)f(e_{2(i-1)}^{'})f(e_{2i-1}^{'}) \text{ for } i = 2,3,\ldots,n-1.$$
$$f^*(w_i^{'}) = f(e_{2i-1}^{'})f(e_{2i-1}^{''}) \text{ for } i = 1,2,\ldots,n-1.$$
$$f^*(w_i^{''}) = f(e_{2i}^{'})f(e_{2i}^{''}) \text{ for } i = 1,2,\ldots,n-1.$$
$$f^*(w_i) = f(e_{2i-1}^{''})f(e_{2i}^{''}) \text{ for } i = 1,2,\ldots,n-1.$$

In view of the above defined labeling pattern, we have
$v_f(1) = \left\{v_1, v_2, v_3, \ldots, v_{\left[\frac{n}{2}\right]}, w_1^{'}, w_2^{'}, \ldots, w_{\left[\frac{n}{2}\right]}^{'}, w_1^{''}, w_2^{''}, \ldots, w_{\left[\frac{n-1}{2}\right]}^{''}, w_1, w_2, w_3, \ldots, w_{\left[\frac{n-1}{2}\right]}\right\}$
and $v_f(0) = \left\{v_{\left[\frac{n}{2}\right]+1}, v_{\left[\frac{n}{2}\right]+2}, \ldots, v_n, w_{\left[\frac{n}{2}\right]+1}^{'}, w_{\left[\frac{n}{2}\right]+2}^{'}, \ldots, w_{n-1}^{'}, w_{\left[\frac{n-1}{2}\right]+1}^{''}, w_{\left[\frac{n}{2}\right]+2}^{''}, \ldots,\right.$
$\left. w_{n-1}^{''}, w_{\left[\frac{n}{2}\right]+1}, w_{\left[\frac{n}{2}\right]+2} \ldots, w_{n-1}\right\}.$
So $v_f(0) = v_f(1)+1 = 2n-2+1 = 2n-1$ and $e_f(0) = \left\lfloor \frac{5n-5}{2} \right\rfloor$, $e_f(1) = \left\lceil \frac{5n-5}{2} \right\rceil$.
Thus $\left|v_f(0) - v_f(1)\right| \leq 1$ and $\left|e_f(0) - e_f(1)\right| \leq 1$.

Thus f admits edge product cordial labeling on PS_n. So the graph PS_n is an edge product cordial graph.

Example 2 An edge product cordial labeling of PS_5 is shown in the following Figure 2. □

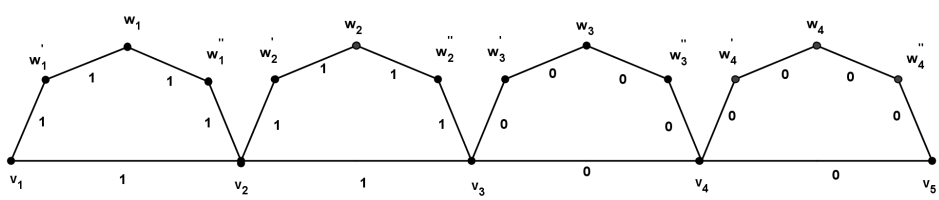

Fig. 2. Edge product cordial labeling of PS_5

Theorem 3 *The graph DPS_n is an edge product cordial graph if and only if n is odd.* □

Proof: Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices and $e_1, e_2, \ldots, e_{n-1}$ be the consecutive edges of the path P_n. To construct DPS_n from the path P_n join v_i to $w_i^{'}$ and $u_i^{'}$ by the edges $e_{2i-1}^{'} = v_i w_i^{'}$ and $f_{2i-1} = v_i u_i^{'}$ respectively for $i = 1,2,\ldots,n-1$. Now join v_{i+1} to $w_i^{''}$ and $u_i^{''}$ by the edge $e_{2i}^{'} = v_{i+1} w_i^{''}$ and $f_{2i}^{'} = v_{i+1} u_i^{''}$ respectively for $i = 1,2,\ldots,n-1$. Now join $w_i^{'}$ and $w_i^{''}$ to a single vertex w_i by the edges $e_{2i-1}^{''} = w_i^{'} w_i$ and $e_{2i}^{''} = w_i^{''} w_i$ and join $u_i^{'}$ and $u_i^{''}$ to a single vertex u_i by the edges $f_{2i-1}^{'} = u_i^{'} u_i$ and

$f'_{2i} = u''_i u_i$ for $i = 1,2,\ldots,n-1$. Thus $|V(DPS_n)| = 7n-6$ and $|E(DPS_n)| = 9n-9$. We consider the following two cases:

case 1: If n is odd, define the mapping $g : E(DPS_n) \to \{0,1\}$ as follows:

$$g(e) = \begin{cases} 1 & \text{if} \quad e = e_i \ \text{ for } \ i = 1,2,\ldots,\dfrac{n-1}{2}; \\ 0 & \text{if} \quad e = e_i \ \text{ for } \ i = \dfrac{n-1}{2}+1, \dfrac{n-1}{2}+2,\ldots,n-1; \\ 1 & \text{if} \quad e = e'_i \ \text{ for } \ i = 1,2,\ldots,n-1; \\ 0 & \text{if} \quad e = e'_i \ \text{ for } \ i = n,n+1,\ldots,2n-2; \\ 1 & \text{if} \quad e = e''_i \ \text{ for } \ i = 1,2,\ldots,n-1; \\ 0 & \text{if} \quad e = e''_i \ \text{ for } \ i = n,n+1,\ldots,2n-2; \\ 1 & \text{if} \quad e = f_i \ \text{ for } \ i = 1,2,\ldots,n-1; \\ 0 & \text{if} \quad e = f_i \ \text{ for } \ i = n,n+1,\ldots,2n-2; \\ 1 & \text{if} \quad e = f'_i \ \text{ for } \ i = 1,2,\ldots,n-1; \\ 0 & \text{if} \quad e = f'_i \ \text{ for } \ i = n,n+1,\ldots,2n-2. \end{cases}$$

The induced vertex labeling function $g^* : V(DPS_n) \to \{0,1\}$ is given by:

$g^*(v_1) = g(e'_1)g(e_1)g(f_1)$.
$g^*(v_n) = g(f_{2n-2})g(e'_{2n-2})g(e_{n-1})$.
$g^*(v_i) = g(e_{i-1})g(e_i)g(e'_{2i-2})g(e'_{2i-1})g(f_{2i-2})g(f_{2i-1})$ for $i = 2,3,\ldots,n-1$.
$g^*(w'_i) = g(e'_{2i-1})g(e''_{2i-1})$ for $i = 1,2,\ldots,n-1$.
$g^*(w''_i) = g(e'_{2i})g(e''_{2i})$ for $i = 1,2,\ldots,n-1$.
$g^*(w_i) = g(e''_{2i-1})g(e''_{2i})$ for $i = 1,2,\ldots,n-1$.
$g^*(u'_i) = g(f_{2i-1})g(f'_{2i-1})$ for $i = 1,2,\ldots,n-1$.
$g^*(u''_i) = g(f_{2i})g(f'_{2i})$ for $i = 1,2,\ldots,n-1$.
$g^*(u_i) = g(f'_{2i-1})g(f'_{2i})$ for $i = 1,2,\ldots,n-1$.

In view of the above defined labeling pattern, we have
$v_f(1) = \Big|\Big\{v_1, v_2, \ldots, v_{\frac{n-1}{2}}, w'_1, w'_2, \ldots, w'_{\frac{n-1}{2}}, w''_1, w''_2, \ldots, w''_{\frac{n-1}{2}}, w_1, w_2, \ldots, w_{\frac{n-1}{2}}, u'_1, u'_2, \ldots, u'_{\frac{n-1}{2}}, u''_1, u''_2, \ldots, u''_{\frac{n-1}{2}}, u_1, u_2, u_3, \ldots, u_{\frac{n-1}{2}}\Big\}\Big|$ and
$v_f(0) = \Big|\Big\{v_{\frac{n-1}{2}+1}, v_{\frac{n-1}{2}+2}, \ldots, v_n, w'_{\frac{n-1}{2}+1}, w'_{\frac{n-1}{2}+2}, \ldots, w'_{n-1}, w''_{\frac{n-1}{2}+1}, w''_{\frac{n-1}{2}+2}, \ldots, w''_{n-1}, w_{\frac{n-1}{2}+1}, w_{\frac{n-1}{2}+2} \ldots, w_{n-1}, u'_{\frac{n-1}{2}+1}, u'_{\frac{n-1}{2}+2}, \ldots, u'_{n-1}, u''_{\frac{n-1}{2}+1}, u''_{\frac{n-1}{2}+2}, \ldots, u''_{n-1}, u_{\frac{n-1}{2}+1}, u_{\frac{n-1}{2}+2} \ldots, u_{n-1}\Big\}\Big|$. So $v_f(0) = v_f(1) + 1 = \dfrac{7n-7}{2} + 1$ and $e_f(0) = e_f(1) = \dfrac{9n-9}{2}$.

Thus $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$. Thus f admits edge product cordial labeling on DPS_n when n is odd. So DPS_n is an edge product cordial graph for odd n.

case 2: If n is even, then in order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\left\lceil \frac{9n-9}{2} \right\rceil$ edges out of $9n-9$ edges. So in this context, the edges with label 0 will give rise to at least $\frac{7n-4}{2}$ vertices with label 0 and at most $\frac{7n-8}{2}$ vertices with label 1 out of $7n-6$ vertices. Therefore $|v_f(0)-v_f(1)| \geq 2$. So DPS_n is not an edge product cordial graph for even n.

Example 3 An edge product cordial labeling of DPS_5 is shown in the following Figure 3. □

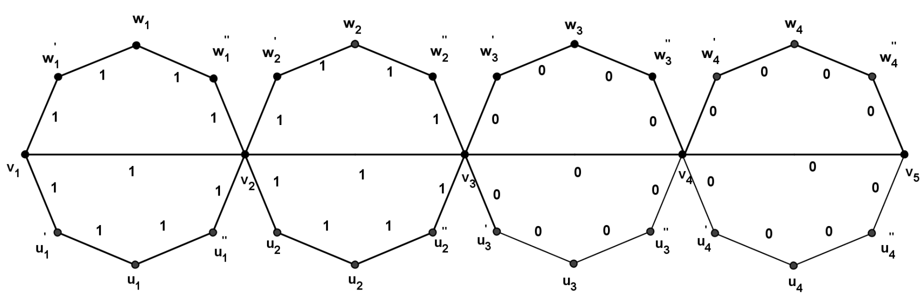

Fig. 3. Edge product cordial labeling of DPS_5

3. Conclusions

In this paper we discussed the edge product cordial labeling for $W_n^{(t)}$ if and only if t is even. Also we have discussed edge product cordial labeling for PS_n and DPS_n if and only if n is odd. Labeling patteren is illustrated by means of examples. To derive similar problems for other graph families is an open area of research.

Acknowledgment

The authors are highly thankful to the anonymous referees for their valuable comments and fruitful suggestions on the first draft of the paper.

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