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THE GENERALIZED DIFFERENTIAL TRANSFORM METHOD FOR SOLUTION OF A FREE VIBRATION LINEAR DIFFERENTIAL EQUATION WITH FRACTIONAL DERIVATIVE DAMPING

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Abstract. In the present paper, the Generalized Differential Transform Method (GDTM) is used for obtaining the approximate analytic solutions of a free vibration linear differential equation of a single-degree-of-freedom (SDOF) system with fractional derivative damping. The fractional derivatives are described in the Caputo sense.

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1. Introduction

Differential equations with fractional order are generalizations of classical differential equations of integer order and have recently been proved to be valuable tools in the modeling of many physical phenomena in various fields of science and engineering. By using fractional derivatives, a lot of work has been done for a better description of considered material properties. Based on enhanced rheological models, Mathematical modeling naturally leads to differential equations of fractional order and to the necessity of the formulation of the initial conditions to such equations. Recently, various analytical and numerical methods have been employed to solve linear and nonlinear fractional differential equations. The differential transform method (DTM) was proposed by Zhou [1] to solve linear and nonlinear initial value problems in electric circuit analysis. This method has been used for solving various types of equations by many authors [2-15]. DTM constructs an analytical solution in the form of a polynomial and different from the traditional higher order Taylor series method. For solving two-dimensional linear and nonlinear partial differential equations of fractional order, DTM is further developed as the

Generalized Differential Transform Method (GDTM) by Momani, Odibat, and Erturk in their papers [16-18].

Recently, Vedat Suat Ertiirka and Shaher Momanib applied the generalized differential transform method to solve fractional integro differential equations [19]. The GDTM is implemented to derive the solution of space-time fractional telegraph equation by Mridula Garg, Pratibha Manohar and Shyam L. Kalla [20]. Manish Kumar Bansal, Rashmi Jain applied generalized differential transform method to solve the fractional order Riccati differential equation [21]. Aysegul Cetinkaya, Onur Kiymaz and Jale Camli applied generalized differential transform method to solve non linear PDE's of fractional order [22].

2. Mathematical preliminaries on fractional calculus

In the present analysis we introduce the following definitions [23, 24].

Definition 1. Let $\alpha \in R^+$ On the usual Lebesgue space $L(a,b)$ integral operator I^α defined by

$$I^\alpha f(x) = \frac{d^{-\alpha} f(x)}{dx^{-\alpha}} = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt \text{ and}$$
$$I^0 f(x) = f(x)$$

is called Riemann-Liouville fractional integral operator of order $\alpha \ge 0$ and $a \le x < b$.

It has the following properties:

I. $I^\alpha f(x)$ exists for any $x \in [a,b]$

II. $I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x)$

III. $I^\alpha I^\beta f(x) = I^\beta I^\alpha f(x)$

IV. $I^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

where $f(x) \in L[a,b]$, $\alpha, \beta \ge 0, \gamma > -1$.

Definition 2. The Riemann-Liouville definition of fractional order derivative is

$${}_0^{RL}D_x^\alpha f(x) = \frac{d^n}{dx^n}\, {}_0I_x^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\,dt,$$

where n is an integer that satisfies $n-1 < \alpha < n$.

Definition 3. A modified fractional differential operator ${}_0^cD_x^\alpha$ proposed by Caputo is given by

$${}_0^cD_x^\alpha f(x) = {}_0 I_x^{n-\alpha} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt,$$

Where $\alpha\left(\alpha \in R^{+}\right)$ is the order of operation and n is an integer that satisfies $n-1<\alpha<n$.

It has the following two basic properties [25]:

I. If $f \in L_{\infty}(a,b)$ or $f \in C[a,b]$ and $\alpha>0$ then ${}_{0}^{c}D_{x}^{\alpha}\,{}_{0}I_{x}^{\alpha} f(x)=f(x)$.

II. If $f \in C^{n}[a,b]$ and if $\alpha>0$ then ${}_{0}I_{x}^{\alpha}\,{}_{0}^{c}D_{x}^{\alpha} f(x)=f(x)-\sum_{k=0}^{n-1} \frac{f^{(k)}\left(0^{+}\right)}{k!} x^{k}$; $n-1<\alpha<n$.

Definition 4. For m being the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha>0$, is defined as [26]

$$D_{t}^{\alpha} u(x,t)=\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} =\begin{cases} \dfrac{\partial^{m} u(x,\xi)}{\partial \xi^{m}} & ; \quad \alpha=m \in N \\ \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{0}^{t}(t-\xi)^{m-\alpha-1} \frac{\partial^{m} u(x,\xi)}{\partial \xi^{m}} d\xi & ; \quad m-1 \leq \alpha<m \end{cases}$$

Relation between Caputo derivative and Riemann-Liouville derivative:

$${}_{0}^{c}D_{x}^{\alpha} f(x)={}_{0}^{RL}D_{t}^{\alpha} f(x)-\sum_{k=0}^{m-1} \frac{f^{(k)}\left(0^{+}\right)}{\Gamma(k-\alpha+1)} x^{k-\alpha} ; \; m-1<\alpha<m$$

Integrating by parts, we get the following formulae as given by [27]

I. $\int_{a}^{b} g(x)\,{}_{a}^{c}D_{x}^{\alpha} f(x) dx=\int_{a}^{b} f(x)\,{}_{x}^{RL}D_{b}^{\alpha} g(x) dx+\sum_{j=0}^{n-1}\left[{}_{x}^{RL}D_{b}^{\alpha+j-n} g(x)\,{}_{x}^{RL}D_{b}^{n-j-1} f(x)\right]_{a}^{b}$

II. For $n=1$, $\int_{a}^{b} g(x)\,{}_{a}^{c}D_{x}^{\alpha} f(x) dx=\int_{a}^{b} f(x)\,{}_{x}^{RL}D_{b}^{\alpha} g(x) dx+\left[{}_{x}I_{b}^{1-\alpha} g(x).f(x)\right]_{a}^{b}$

3. Generalized one dimensional differential transform method

Generalized differential transform of a function $y(x)$ in one variable is denoted by $Y_{\alpha}(k)$ and defined as follows [16-18]:

$$Y_{\alpha}(k)=\frac{1}{\Gamma(\alpha k+1)}\left[\left(D_{x_{0}}^{\alpha}\right)^{k} y(x)\right]_{x=x_{0}} \tag{1}$$

where $\alpha \in(0,1]$ and $\left(D_{x_{0}}^{\alpha}\right)^{k}=D_{x_{0}}^{\alpha}, D_{x_{0}}^{\alpha}, \ldots.., D_{x_{0}}^{\alpha}$ (k-times).

and the inverse generalized differential transform of $Y_\alpha(k)$ is given by

$$y(x)=\sum_{k=0}^{\infty}Y_\alpha(k)(x-x_0)^{\alpha k} \tag{2}$$

It has the following properties:

I. If $u(x)=v(x)\pm w(x)$ then $U_\alpha(k)=V_\alpha(k)\pm W_\alpha(k)$

II. If $u(x)=av(x)$; $a\in R$ then $U_\alpha(k)=aV_\alpha(k)$

III. If $U(x)=v(x)w(x)$ then $U_\alpha(x)=\sum_{r=0}^{k}V_\alpha(r)W_\alpha(k-r)$

IV. If $u(x)=(x-x_0)^{n\alpha}$ then $U_\alpha(k)=\delta(k-n)$

V. If $u(x)=D_{x_0}^{\alpha}v(x)$; $0<\alpha\le 1$ then $U_\alpha(k)=\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}V_\alpha(k+1)$

VI. If $u(x)=x^{\lambda}f(x)$ where $\lambda>-1$, $f(x)$ has the generalized Taylor series expansion $f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^{n\alpha}$ and

a. $\beta<\lambda+1$ and α is arbitrary or

b. $\beta\ge\lambda+1,\alpha$ arbitrary and $a_n=0$ for $n=0,1,2,....m-1$, where $m-1<\beta\le m$.
Then (1) becomes

$$U_\alpha(k)=\frac{1}{\Gamma(\alpha k+1)}\left[D_{x_0}^{\alpha k}u(x)\right]_{x_0}$$

VII. If $u(x)=D_{x_0}^{\gamma}f(x)$, $m-1<\gamma\le m$ and the function $f(x)$ satisfies the conditions given in (VI) then $U_\alpha(k)=\frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}F_\alpha\left(k+\frac{\gamma}{\alpha}\right)$

where $U_\alpha(k)$, $V_\alpha(k)$, $W_\alpha(k)$ and $F_\alpha(k)$ are the differential transformations of the functions $u(x)$, $v(x)$, $w(x)$ and $f(x)$ respectively and

$$\delta(k-n)=\begin{cases}1 & ; \quad k=n\\ 0 & ; \quad k\ne n\end{cases}$$

4. Solution of the free vibration linear differential equation of single-degree-of-freedom (SDOF) system with fractional derivative damping

In this section, we consider the free vibration linear differential equation of single-degree-of-freedom (SDOF) system with fractional derivative damping

$$m\frac{d^2x(t)}{dt^2}+c\frac{d^\alpha x(t)}{dt^\alpha}+kx(t)=0 \tag{3}$$

subject to initial conditions $x(0)=p$ (constant) and $x'(0)=q$ (constant),

where $\frac{d^\alpha}{dt^\alpha}$ is the fractional differential operator (Caputo derivative) of order $0<\alpha\le 1$ and *m*, *c*, *k* are the mass, damping and stiffness coefficient respectively.

Applying generalized one-dimensional differential transform (1) with $t_0=0$ on (3) we obtain

$$X_\alpha(h)=-\frac{\Gamma\left(\alpha\left(h-\frac{2}{\alpha}\right)+1\right)}{m\Gamma\left(\alpha\left(h-\frac{2}{\alpha}\right)+3\right)}\left[c\frac{\Gamma\left(\alpha\left(h-\frac{2}{\alpha}+1\right)+1\right)}{\Gamma\left(\alpha\left(h-\frac{2}{\alpha}\right)+1\right)}X_\alpha\left(h-\frac{2}{\alpha}+1\right)+kX_\alpha\left(h-\frac{2}{\alpha}\right)\right] \tag{4}$$

with $X_\alpha(0)=p$ and $X_\alpha\left(\frac{1}{\alpha}\right)=q.$ (5)

Taking $\alpha=1/2$, then (4) and (5) becomes

$$X_{1/2}(h)=-\frac{\Gamma\left(\frac{1}{2}h-1\right)}{m\Gamma\left(\frac{1}{2}h+1\right)}\left[c\frac{\Gamma\left(\frac{1}{2}h-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}h-1\right)}X_{1/2}(h-3)+kX_{1/2}(h-4)\right] \tag{6}$$

with $X_{1/2}(0)=p$ and $X_{1/2}(2)=q$ (7)

Now utilizing the recurrence relation (6) and the initial condition (7), after a little simplification we obtain the following values of $X_{1/2}(k)$ for $k=0,1,2,......$

$$X_{1/2}(1)=0;\ X_{1/2}(3)=0;\ X_{1/2}(4)=-\frac{kp}{m}\frac{1}{\Gamma(3)};\ X_{1/2}(5)=-\frac{cq}{m}\frac{1}{\Gamma(7/2)};$$

$$X_{1/2}(6)=-\frac{kq}{m}\frac{1}{\Gamma(4)};\ X_{1/2}(7)=\frac{ckp}{m^2}\frac{1}{\Gamma(9/2)};\ X_{1/2}(8)=\frac{c^2q+k^2p}{m^2}\frac{1}{\Gamma(5)};$$

$$X_{1/2}(9)=\frac{2ckq}{m^2}\frac{1}{\Gamma(11/2)};\ X_{1/2}(10)=\left(\frac{k^2q}{m^3}-\frac{c^2kp}{m^3}\right)\frac{1}{\Gamma(6)};$$

$$X_{1/2}(11) = -\left(\frac{c\left(c^2q + k^2p\right)}{m^3} + \frac{ck^2p}{m^3}\right)\frac{1}{\Gamma\left(13/2\right)};$$

$$X_{1/2}(12) = -\left(\frac{2kqc^2}{m^3} + \frac{k\left(c^2q + k^2p\right)}{m^3}\right)\frac{1}{\Gamma(7)}$$

and so on.

Now, from (2), we have

$$x(t) = \sum_{h=0}^{\infty} X_{1/2}(h)\, t^{h/2} \tag{8}$$

Using the above values of $X_{1/2}(k)$; $k = 0,1,2,.....$ in (8) the solution of (3) is obtained as

$$x(t) = p + qt - \frac{kp}{m}\frac{1}{\Gamma(3)}t^2 - \frac{cq}{m}\frac{1}{\Gamma\left(7/2\right)}t^{5/2} - \frac{kq}{m}\frac{1}{\Gamma(4)}t^3 + \frac{ckp}{m^2}\frac{1}{\Gamma\left(9/2\right)}t^{7/2}$$
$$+ \frac{c^2q + k^2p}{m^2}\frac{1}{\Gamma(5)}t^4 + \frac{2ckq}{m^2}\frac{1}{\Gamma\left(11/2\right)}t^{9/2} + \left(\frac{k^2q}{m^3} - \frac{c^2kp}{m^3}\right)\frac{1}{\Gamma(6)}t^5$$
$$-\left(\frac{c\left(c^2q + k^2p\right)}{m^3} + \frac{ck^2p}{m^3}\right)\frac{1}{\Gamma\left(13/2\right)}t^{11/2} - \left(\frac{2kqc^2}{m^3} + \frac{k\left(c^2q + k^2p\right)}{m^3}\right)\frac{1}{\Gamma(7)}t^6 + ... \tag{9}$$

Taking $\alpha = 1/3$, then (4) and (5) becomes

$$X_{1/3}(h) = -\frac{\Gamma\left(\frac{1}{3}h - 1\right)}{m\Gamma\left(\frac{1}{3}h + 1\right)}\left[c\frac{\Gamma\left(\frac{1}{3}h - \frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}h - 1\right)}X_{1/3}(h-5) + kX_{1/3}(h-6)\right] \tag{10}$$

with

$$X_{1/3}(0) = p \text{ and } X_{1/3}(3) = q \tag{11}$$

Now utilizing the recurrence relation (10) and the initial condition (11), after a little simplification we obtain the following values of $X_{1/3}(k)$ for $k = 0,1,2,......$

$$X_{1/3}(2)=0;\ X_{1/3}(1)=0;\ X_{1/3}(4)=0;\ X_{1/3}(5)=0;\ X_{1/3}(6)=-\frac{kp}{m}\frac{1}{\Gamma(3)};$$

$$X_{1/3}(7)=0;\ X_{1/3}(8)=-\frac{cq}{m}\frac{1}{\Gamma\left(11/3\right)};\ X_{1/3}(9)=-\frac{kq}{m}\frac{1}{\Gamma(4)};\ X_{1/3}(10)=0;$$

$$X_{1/3}(11)=\frac{ckp}{m^2}\frac{1}{\Gamma\left(14/3\right)};\ X_{1/3}(12)=\frac{k^2p}{m^2}\frac{1}{\Gamma(5)};\ X_{1/3}(13)=\frac{c^2q}{m^2}\frac{1}{\Gamma\left(16/3\right)};$$

$$X_{1/3}(14)=\frac{2ckq}{m^2}\frac{1}{\Gamma\left(17/3\right)};\ X_{1/3}(15)=\frac{k^2q}{m^2}\frac{1}{\Gamma(6)};\ X_{1/3}(16)=-\frac{c^2kp}{m^3}\frac{1}{\Gamma\left(19/3\right)}$$

and so on.

Now, from (2), we have

$$x(t)=\sum_{h=0}^{\infty}X_{1/3}(h)t^{h/3} \tag{12}$$

Using the above values of $X_{1/3}(k);\ k=0,1,2,.....$ in (12) the solution of (3) is obtained as

$$x(t)=p+qt-\frac{kp}{m}\frac{1}{\Gamma(3)}t^2-\frac{cq}{m}\frac{1}{\Gamma\left(11/3\right)}t^{8/3}-\frac{kq}{m}\frac{1}{\Gamma(4)}t^3+\frac{ckp}{m^2}\frac{1}{\Gamma\left(14/3\right)}t^{11/3}$$
$$+\frac{k^2p}{m^2}\frac{1}{\Gamma(5)}t^4+\frac{c^2q}{m^2}\frac{1}{\Gamma\left(16/3\right)}t^{13/3}+\frac{2ckq}{m^2}\frac{1}{\Gamma\left(17/3\right)}t^{14/3}+\frac{k^2q}{m^2}\frac{1}{\Gamma(6)}t^5$$
$$-\frac{c^2kp}{m^3}\frac{1}{\Gamma\left(19/3\right)}t^{16/3}+........... \tag{13}$$

Taking $\alpha=1/4$, then (4) and (5) becomes

$$X_{1/4}(h)=-\frac{\Gamma\left(\frac{1}{4}h-1\right)}{m\Gamma\left(\frac{1}{4}h+1\right)}\left[c\frac{\Gamma\left(\frac{1}{4}h-\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}h-1\right)}X_{1/4}(h-7)+kX_{1/4}(h-8)\right] \tag{14}$$

with $X_{1/4}(0)=p$ and $X_{1/4}(4)=q$ (15)

Now utilizing the recurrence relation (14) and the initial condition (15), after a little simplification we obtain the following values of $X_{1/4}(k)$ for $k=0,1,2,......$

$$X_{1/4}(2)=0;\ X_{1/4}(1)=0;\ X_{1/4}(3)=0;\ X_{1/4}(5)=0;\ X_{1/4}(6)=0;$$

$$X_{1/4}(7)=0;\ X_{1/4}(8)=-\frac{kp}{m}\frac{1}{\Gamma(3)};\ X_{1/4}(9)=0;\ X_{1/4}(10)=0;$$

$$X_{1/4}(11)=-\frac{cq}{m}\frac{1}{\Gamma\left(15/4\right)};\ X_{1/4}(12)=-\frac{kq}{m}\frac{1}{\Gamma(4)};\ X_{1/4}(13)=0;\ X_{1/4}(14)=0;$$

$$X_{1/4}(15)=\frac{ckp}{m^2}\frac{1}{\Gamma\left(19/4\right)};\ X_{1/4}(16)=\frac{k^2p}{m^2}\frac{1}{\Gamma(5)};\ X_{1/4}(17)=0;$$

$$X_{1/4}(18)=\frac{c^2q}{m^2}\frac{1}{\Gamma\left(22/4\right)};\ X_{1/4}(19)=\frac{2ckq}{m^2}\frac{1}{\Gamma\left(23/4\right)};\ X_{1/4}(20)=\frac{k^2q}{m^2}\frac{1}{\Gamma(6)};$$

$$X_{1/4}(21)=0;\ X_{1/4}(22)=-\frac{c^2kp}{m^3}\frac{1}{\Gamma\left(26/4\right)}$$

and so on.

Now, from (2), we have

$$x(t)=\sum_{h=0}^{\infty}X_{1/4}(h)t^{h/4} \tag{16}$$

Using the above values of $X_{1/4}(k);\ k=0,1,2,.....$ in (16) the solution of (3) is obtained as

$$x(t)=p+qt-\frac{kp}{m}\frac{1}{\Gamma(3)}t^2-\frac{cq}{m}\frac{1}{\Gamma\left(15/4\right)}t^{11/4}-\frac{kq}{m}\frac{1}{\Gamma(4)}t^3+\frac{ckp}{m^2}\frac{1}{\Gamma\left(19/4\right)}t^{15/4}$$
$$+\frac{k^2p}{m^2}\frac{1}{\Gamma(5)}t^4+\frac{c^2q}{m^2}\frac{1}{\Gamma\left(11/2\right)}t^{9/2}+\frac{2ckq}{m^2}\frac{1}{\Gamma\left(23/4\right)}t^{19/4}+\frac{k^2q}{m^2}\frac{1}{\Gamma(6)}t^5$$
$$-\frac{c^2kp}{m^3}\frac{1}{\Gamma\left(13/2\right)}t^{11/2}+....... \tag{17}$$

Taking $\alpha=1/5$, then (4) and (5) becomes

$$X_{1/5}(h)=-\frac{\Gamma\left(\frac{1}{5}h-1\right)}{m\Gamma\left(\frac{1}{5}h+1\right)}\left[c\frac{\Gamma\left(\frac{1}{5}h-\frac{4}{5}\right)}{\Gamma\left(\frac{1}{5}h-1\right)}X_{1/5}(h-9)+kX_{1/5}(h-10)\right] \tag{18}$$

with $X_{1/5}(0) = p$ and $X_{1/5}(5) = q$ (19)

Now utilizing the recurrence relation (18) and the initial condition (19), after a little simplification we obtain the following values of $X_{1/5}(k)$ for $k = 0,1,2,......$

$$X_{1/5}(2) = 0;\ X_{1/5}(1) = 0;\ X_{1/5}(3) = 0;\ X_{1/5}(4) = 0;\ X_{1/5}(6) = 0;$$

$$X_{1/5}(7) = 0;\ X_{1/5}(8) = 0;\ X_{1/5}(9) = 0;\ X_{1/5}(10) = -\frac{kp}{m}\frac{1}{\Gamma(3)};\ X_{1/5}(11) = 0;$$

$$X_{1/5}(12) = 0;\ X_{1/5}(13) = 0;\ X_{1/5}(14) = -\frac{cq}{m}\frac{1}{\Gamma\left(19/5\right)};\ X_{1/5}(15) = -\frac{kq}{m}\frac{1}{\Gamma(4)};$$

$$X_{1/5}(16) = 0;\ X_{1/5}(17) = 0;\ X_{1/5}(18) = 0;\ X_{1/5}(20) = \frac{k^2 p}{m^2}\frac{1}{\Gamma(5)};$$

$$X_{1/5}(21) = 0;\ X_{1/5}(22) = 0;\ X_{1/5}(23) = \frac{c^2 q}{m^2}\frac{1}{\Gamma\left(28/5\right)};$$

$$X_{1/5}(24) = \frac{2ckq}{m^2}\frac{1}{\Gamma\left(29/5\right)};\ X_{1/5}(25) = \frac{k^2 q}{m^2}\frac{1}{\Gamma(6)};\ X_{1/5}(26) = 0;$$

$$X_{1/5}(27) = 0;\ X_{1/5}(28) = -\frac{c^2 kp}{m^3}\frac{1}{\Gamma\left(33/5\right)}$$

and so on.

Now, from (2), we have

$$x(t) = \sum_{h=0}^{\infty} X_{1/5}(h)\, t^{h/5} \tag{20}$$

Using the above values of $X_{1/5}(k);\ k = 0,1,2,.....$ in (20) the solution of (3) is obtained as

$$x(t) = p + qt - \frac{kp}{m}\frac{1}{\Gamma(3)}t^2 - \frac{cq}{m}\frac{1}{\Gamma\left(19/5\right)}t^{14/5} - \frac{kq}{m}\frac{1}{\Gamma(4)}t^3 + \frac{ckp}{m^2}\frac{1}{\Gamma\left(24/5\right)}t^{19/5}$$
$$+\frac{k^2 p}{m^2}\frac{1}{\Gamma(5)}t^4 + \frac{c^2 q}{m^2}\frac{1}{\Gamma\left(28/5\right)}t^{23/5} + \frac{2ckq}{m^2}\frac{1}{\Gamma\left(29/5\right)}t^{24/5} + \frac{k^2 q}{m^2}\frac{1}{\Gamma(6)}t^5$$
$$-\frac{c^2 kp}{m^3}\frac{1}{\Gamma\left(33/5\right)}t^{28/5} + ... \tag{21}$$

5. Conclusions

In the present study, we have applied the Generalized Differential Transform Method (GDTM) to find the approximate analytic solution of the free vibration linear differential equation of single-degree-of-freedom (SDOF) system with fractional derivative damping. It may be concluded that the GDTM is a reliable technique to handle linear and nonlinear fractional differential equations. The GDTM provides more realistic series solutions compared with other approximate methods.

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