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ON THE USE OF MOHAND INTEGRAL TRANSFORM FOR SOLVING FRACTIONAL-ORDER CLASSICAL CAPUTO DIFFERENTIAL EQUATIONS

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Abstract. In this research study, a newly devised integral transform called the Mohand transform has been used to find the exact solutions of fractional-order ordinary differential equations under the Caputo type operator. This transform technique has successfully been employed in existing literature to solve classical ordinary differential equations. Here, a few significant and practically-used differential equations of the fractional type, particularly related with kinetic reactions from chemical engineering, are under consideration for the possible outcomes via the Mohand integral transform. A new theorem has been proposed whose proof, provided in the present study, helped to get the exact solutions of the models under investigation. Upon comparison, the obtained results would agree with results produced by other existing well-known integral transforms including Laplace, Fourier, Mellin, Natural, Sumudu, Elzaki, Shehu and Aboodh.

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1. Introduction

The area of fractional calculus has currently become a burning area of study due to its numerous applications in almost every field of science, engineering and finance. Whether it be quantum mechanics, fluid dynamics, bio-medical, epidemiology, clinical biochemistry, chemical kinetics, statistical field theory, continuous random fields, electromagnetism, groundwater study, aerospace engineering, actuaries and many more as can be seen in recently conducted research studies [1–14] and most of the references cited therein.

Classical calculus or integer-order derivatives and integration has some limitations wherein the major limitation comes from locality of the operators used in this type of calculus. By locality, we mean that the future state of any physical or natural system under investigation depends only on its current state. On the other hand, fractional--order operators take into account the entire history of the systems' behavior to predict its future, thereby having non-locality embedded in the very nature of such operators. Thus, the classical calculus is not suitable in situations where mathematical models follow non-Markovian behavior.

The non-locality characteristic is an excellent approach for modeling various physical and natural phenomena and realities since most of the models of this type are memory affected and need careful treatment with these non-local operators to get more accurate and realistic results, and thus the non-Markovian nature is easily captured. Indeed, this has been proved in most recently-published works such as modeling, with the help of real data, of epidemiogical tuberculosis virus infection with the classical Caputo [15] is proved to have more accurate results than the existing classical model which depends solely upon first order ordinary differential equations. Again, with the help of real data application, the fractional-order version of the mathematical model of blood ethanol concentration is found to be much better than the classical one [16]. However, non-locality poses a great challenge to numerical analysts to design codes having computational effectiveness in terms of both time complexity and machine memory.

The natural occurrence of such memory dependent and hereditary properties in many mathematical models motivated various scientists to design methods to get their exact and approximate solutions. As far as exact solutions are concerned, the most commonly-used integral transform called the Laplace transform plays a vital role to solve fractional-order initial value problems. Some others include Fourier, Mellin, Sumudu, Natural, Shehu, Elzaki and Aboodh.

The ubiquity of fractional-order models in many fields and emergence of new effective integral transforms led us to try our hands on some existing problems of practical importance. In this connection, an integral transform called the Mohand transform has been applied in the present study to find exact solutions of fractional--order initial value problems under the classical Caputo fractional-order derivative operator.

2. Mathematical preliminaries

In this section, some basic concepts related to the Caputo fractional calculus are listed which are considered necessary to revisit for the convenience of the reader.

Definition 1 [17] The Riemann-Liouville α-integral of arbitrary real order $\alpha > 0$ of a function $g(t)$ is defined by the following integral equation:

$$J_{0,t}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\nu)^{\alpha-1}g(\nu)d\nu,\ t>0.$$

Definition 2 [17] The fractional-order derivative of $g(t)$ under the classical Caputo definition with $\alpha > 0$ is given by

$$^{C}D_{0,t}^{\alpha}g(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\nu)^{m-\alpha-1}g^{(m)}(\nu)d\nu, t>0, m-1<\alpha\leq m,\ m\in\mathbb{N}. \tag{1}$$

If $\alpha = 1$, the Caputo non-integer order derivative reduces to the ordinary first order derivative from the classical calculus.

Definition 3 [18] Let $g(t)$ be piece-wise defined, of exponential order P and belong to the set A by:

$$A = \{g(t) : \exists P, a_0, a_1 > 0, |g(t)| < P\exp\left(\frac{|t|}{a_i}\right) \textit{when}\, t\in(-1)^i\times[0,\infty)\}, \tag{2}$$

then the Mohand integral transform for the function $g(t)$ is defined as follows:

$$M\Big(g(t)\Big) = M(s) = s^2\int_0^{\infty} g(t)\exp(-st)dt, \tag{3}$$

where $t\geq 0$, $s\in[a_0,a_1]$. □

Lemma 1 *[17] The relation between the fractional-order classical Caputo derivative operator and the Riemann-Liouville α-integral is defined by the following:*

$$^{C}D_{0,t}^{\alpha}g(t) = J_{0,t}^{m-\alpha}\Big(D^m g(t)\Big),\ (m-1)\alpha < m\in Z^{+}. \tag{4}$$

3. Research methodology

This section is devoted to the discussion of the steps taken into the direction of solving fractional-order initial value problems under the classical Caputo fractional--order derivative operator. In order to serve the purpose, an integral transform technique (so far not tested on such problems in literature) called the Mohand transform has been employed.

The Mohand transform was first introduced in [18] and later further explored in [19]. Its physical applications are also explained in [20] in connection with the classical mathematical models which depend upon first order linear ordinary differential equations. This integral transform determines the exact solution of fractional--order problems with the help of a new theorem that was proposed and proved in the present research study.

Comparison with some well-known existing integral transforms (Laplace, Fourier, Mellin, Sumudu, Natural, Elzaki, Aboodh) can be carried out in order to be sure about the exact solutions obtained in the current work. The Mohand integral transform is defined by the equation (3) and will be used throughout the present research work.

3.1. Basic properties of the Mohand transform

Some of the basic properties of the Mohand transform are listed below [21–24] which are necessary to be revisited to comprehend the rest of the analysis.

Linearity

The linearity property of the Mohand integral transform states that

$$k_1 g_1(t) \pm k_2 g_2(t) \overset{\mathrm{M}}{\longleftrightarrow} k_1 G_1(s) \pm k_2 G_2(s) \tag{5}$$

Shifting Theorem

If $\mathrm{M}(g(t)) = G(s)$, when $s > k$ then,

$$M\Big(\exp(kt) g(t)\Big) = s^2 G(s-k). \tag{6}$$

Powers of t

If $g(t) = t^k$, when $s > k$ and $k+1 > 0$ then,

$$M\Big(t^k\Big) = \frac{\Gamma(k+1)}{s^{k-1}}. \tag{7}$$

Exponentials

If $g(t) = \exp(kt)$, when $s > k$ then,

$$M\Big(\exp(kt)\Big) = \frac{s^2}{s-k}. \tag{8}$$

Circular and Hyperbolic Functions

$$\begin{aligned} M\Big(\sin(kt)\Big) &= \frac{s^2}{s^2+k^2},\ M\Big(\cos(kt)\Big) = \frac{s^3}{s^2+k^2}, \\ M\Big(\sinh(kt)\Big) &= \frac{s^2}{s^2-k^2},\ M\Big(\cosh(kt)\Big) = \frac{s^3}{s^2-k^2}. \end{aligned} \tag{9}$$

$\mathrm{N}^{th}-$ Order Derivative

If $\mathrm{M}(g(t)) = G(s)$, then,

$$M\Big(\frac{d^n y(t)}{dt^n}\Big) = s^n G(s) - \sum_{j=0}^{n-1} \frac{y^{(j)}(0)}{s^{j-n-1}},\, n \in \mathbb{N}. \tag{10}$$

$\mathrm{N}^{th}-$ Order Integral

If $\mathrm{M}(g(t)) = G(s)$, then,

$$M\Big(I^n(t)\Big) = \frac{1}{s^n}G(s), \tag{11}$$

where $I^n(t) = \int_0^t \dots \int_0^t y(t)(dt)^n$, and $n \in \mathbb{N}$.

The last two properties will assist us in proving a theorem stated as:

Theorem 1 *Let $M\Big(g(t)\Big) = G(s)$ be the Mohand transform of a piece-wise continuous and exponential order function $g(t)$. The Mohand transform for the fractional--order derivative of the function $g(t)$ under the classical Caputo fractional-order derivative operator of order $\alpha > 0$ is devised as:*

$$M\Big({}^C D^{\alpha}_{0,t} y(t)\Big) = s^{\alpha}G(s) - \sum_{j=0}^{n-1} \frac{y^{(j)}(0)}{s^{j-\alpha-1}}. \tag{12}$$

PROOF In order to get the proof for the existence of the above relation, we use the lemma 1 defined above. Using this lemma, we obtain

$${}^C D^{\alpha}_{0,t} g(t) = J^{n-\alpha}_{0,t}\Big(D^n g(t)\Big),\ n \in \mathbb{N},\ \alpha > 0. \tag{13}$$

Further, suppose that $D^n y(t) = h(t)$ exists and belongs to $C^n(0,\infty)$. Then

$${}^C D^{\alpha}_{0,t} g(t) = J^{n-\alpha}_{0,t}\Big(h(t)\Big),\ n \in \mathbb{N},\ \alpha > 0. \tag{14}$$

Taking the Mohand transform on both sides, we obtain

$$M\Big({}^C D^{\alpha}_{0,t} g(t)\Big) = M\Big(J^{n-\alpha}_{0,t}\Big(h(t)\Big)\Big). \tag{15}$$

Now, using the Mohand transform for integer-order integrals, one can obtain the following

$$M\Big({}^C D^{\alpha}_{0,t} g(t)\Big) = \frac{1}{s^{n-\alpha}} M\Big(h(t)\Big). \tag{16}$$

Further, upon using the Mohand transform of integer-order derivatives, we obtain

$$\begin{aligned} M\Big({}^C D^{\alpha}_{0,t} g(t)\Big) &= \frac{1}{s^{n-\alpha}}\left(s^n G(s) - \sum_{j=0}^{n-1} \frac{y^{(j)}(0)}{s^{j-n-1}}\right) \\ M\Big({}^C D^{\alpha}_{0,t} y(t)\Big) &= s^{\alpha}G(s) - \sum_{j=0}^{n-1} \frac{y^{(j)}(0)}{s^{j-\alpha-1}}. \end{aligned} \tag{17}$$

With this, we complete the proof to obtain the Mohand integral transform for the fractional-order derivatives under the classical Caputo fractional-order derivative operator. ■

4. Results and discussion

In this part of the section, we employ the Mohand integral transform technique to solve some practically important linear fractional-order initial value problems under Caputo's type fractional-order derivative operator. In this connection, the above--proposed theorem will be used to serve the purpose of the present study.

Example 1 Consider the following Bagley-Torvik type in-homogeneous fractional--order initial value problem under the classical Caputo's type fractional-order derivative operator:

$$\frac{d^2y(t)}{dt^2} + ^C\left(\frac{d^\alpha y(t)}{dt^\alpha}\right) + y(t) = 1+t,\ y(0) = y'(0) = 1,\ \alpha = 1.5. \tag{18}$$

The Bagley-Torvik fractional-order differential equation is substantial in various fields of applied sciences as it is capable enough to well describe the behavior of real materials. The $3/2$ – fractional-order equation, such as the one given above, is used to model the frequency dependent damping materials. The motion of both real physical systems and rigid bodies immersed in Newtonian fluid can be described by this type of fractional-order equation. We apply the Mohand integral transform on both sides of Eq. (18) to obtain the following

$$M\left(\frac{d^2y(t)}{dt^2}\right) + M\left(^C\left(\frac{d^\alpha y(t)}{dt^\alpha}\right)\right) + M\Big(y(t)\Big) = M(1+t). \tag{19}$$

Employing the Mohand integral transform for the classical (10) and fractional-order derivatives (12) of a function, one obtains the following

$$s^2Y(s) - \left(\frac{1}{s^{-3}} + \frac{1}{s^{-2}}\right) + s^\alpha Y(s) - \left(\frac{1}{s^{-\alpha-1}} + \frac{1}{s^{-\alpha}}\right) + Y(s) = s+1. \tag{20}$$

We have chosen $n = 2$ as the given value of $\alpha \in (1,2)$. Further simplification of the above step yields the following

$$Y(s)(s^\alpha + s^2 + 1) = s^3 + s^2 + s^{1+\alpha} + s^\alpha + s + 1. \tag{21}$$

$$Y(s) = 1+s. \tag{22}$$

Finally, application of the inverse Mohand integral transform gives

$$y(t) = t+1. \tag{23}$$

Example 2 A first order rate equation is a reaction that depends on the concentration of only one reactant considering other reactants (if present) of order zero. The rate law of such an equation is formulated as

$$\frac{d}{dt}[C_S(t)] = -k[C_S(t)]. \tag{24}$$

Fractionalizing the above equation under the Caputo type fractional derivative operator, one obtains

$$^{C}\left(\frac{d^{\alpha}}{dt^{\alpha}}[C_S(t)]\right) = -k^{\alpha}[C_S(t)], \tag{25}$$

where the initial concentration is given as $C_S(0) = a_0 > 0$. Using the Mohand integral transform on both sides, one obtains

$$\begin{aligned} &M[C_S(t)](s^{\alpha}+k^{\alpha}) - a_0 s^{1+\alpha} = 0, \\ &M[C_S(t)] = a_0 s^2 \frac{s^{\alpha}}{s(s^{\alpha}+k^{\alpha})}. \end{aligned} \tag{26}$$

Upon using the inverse Mohand integral transform, one obtains

$$[C_S(t)] = a_0 E_{\alpha}(-k^{\alpha}t^{\alpha}), \tag{27}$$

where $E_{\alpha}(\cdot)$ is the one-parameter Mittag-Leffler function defined to be $E_{\alpha}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+1)}$. In the above settings, let us suppose that $[C_S(t)](t)$ be the concentration of alcohol in the stomach of a human body at any time t (in minutes) whose solution from [] is obtained in Table 1.

Table 1. Concentration of alcohol in the stomach under classical ($\alpha = 1$) case

Time [min]	0	10	20	30	45
$[C_S]$ [mg/l]	261.7210	85.4404	27.8925	9.1057	1.6984
Time [min]	80	90	110	170	–
$[C_S]$ [mg/l]	0.0338	0.0110	0.0012	0.0000	–

Where the initial concentration of alcohol in the stomach and the rate constant are respectively given as $a_0 = 261.7210$ and $k = 0.111946$. These values are computed for the classical version of the model, that is, when $\alpha = 1$. In order to compute the best values for these parameters for the fractional-order version of the rate law equation under the Caputo type fractional-order derivative operator, we have used the optimization technique thereby getting $a_0 = 991.085$ and $k = 0.0287362$.

Further, it can be observed from Table 2 that the concentration of alcohol in the stomach of a human body takes a substantially long time period to vanish for smaller values of the fractional-order parameter α. This situation conforms the practical experience in real life cases where it is commonly observed to be the same behavior that is clearly depicted by the fractional-order model of first order rate equation in the chemical reactions theory.

It has been observed from Table 2 that the fractional-order parameter α is inversely proportional to the amount of concentration of alcohol when time moves from 0 to 170 minutes leading the concentration to be continuously decreasing. The major reason for non-vanishing behavior of a concentration of alcohol in the fractional-

-order sense is due to the fact that the initially, alcohol enters into the bloodstream wherein 20% of it is absorbed in the stomach whereas the remaining goes in the small intestine with a negligible amount being absorbed in rest of the digestive tract thereby taking a long time period to be completely eliminated from a human's stomach [25]. This peculiar behavior is well captured with the help of fractional-order parameter α taken into consideration in this particular example thus revealing the practical importance of fractional calculus over the existing classical one.

Table 2. Behavior of concentration of alcohol in the stomach under varying values of α

t	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 0.95$
0	9.9109E+02	9.9109E+02	9.9109E+02	9.9109E+02	9.9109E+02	9.9109E+02
10	5.9233E+02	6.1843E+02	6.4729E+02	6.7840E+02	7.1085E+02	7.2724E+02
20	4.9948E+02	5.0451E+02	5.1234E+02	5.2357E+02	5.3869E+02	5.4777E+02
30	4.4395E+02	4.3494E+02	4.2717E+02	4.2130E+02	4.1812E+02	4.1782E+02
45	3.8932E+02	3.6665E+02	3.4336E+02	3.1955E+02	2.9556E+02	2.8366E+02
80	3.1611E+02	2.7746E+02	2.3705E+02	1.9438E+02	1.4879E+02	1.2465E+02
90	3.0202E+02	2.6084E+02	2.1808E+02	1.7325E+02	1.2572E+02	1.0068E+02
110	2.7885E+02	2.3403E+02	1.8829E+02	1.4133E+02	9.2763E+01	6.7704E+01
170	2.3256E+02	1.8284E+02	1.3489E+02	8.9224E+01	4.6480E+01	2.6454E+01

Example 3 Consider a chemical reaction $A \overset{k}{\leftrightarrow} B$ carried out in a batch reactor. The governing system of differential equations for this type of experimental study is formulated as follows:

$$\begin{aligned} \frac{d}{dt}[C_A(t)] &= -k[C_A(t)], \\ \frac{d}{dt}[C_B(t)] &= k[C_A(t)]. \end{aligned} \tag{28}$$

The initial concentration for the species A is found to be $C_A(0) = 1$ mol/m^3 and for B is $C_B(0) = 0$ mol/m^3 with the rate constant k $[s^{-1}]$. □

Fractionalizing the above equation under the Caputo type fractional-order derivative operator, one obtains

$$\begin{aligned} {}^{C}\left(\frac{d^\alpha}{dt^\alpha}[C_A(t)]\right) &= -k^\alpha[C_A(t)], \\ {}^{C}\left(\frac{d^\alpha}{dt^\alpha}[C_B(t)]\right) &= k^\alpha[C_A(t)]. \end{aligned} \tag{29}$$

Using the similar steps as in the previous example, one obtains

$$\begin{aligned} [C_A(t)] &= E_\alpha(-k^\alpha t^\alpha), \\ [C_B(t)] &= 1 - E_\alpha(-k^\alpha t^\alpha). \end{aligned} \tag{30}$$

Table 3. Concentration of the species A under varying values of α with $k = 1\ s^{-1}$

t	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
0	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00	1.0000E+00
20	1.2321E-01	7.8378E-02	4.5195E-02	2.2381E-02	8.0369E-03	2.0612E-09
40	8.8131E-02	5.1066E-02	2.6840E-02	1.2185E-02	4.0489E-03	1.3035E-17
60	7.2244E-02	3.9780E-02	1.9913E-02	8.6412E-03	2.7549E-03	5.4777E-18
80	6.2691E-02	3.3339E-02	1.6148E-02	6.7953E-03	2.1049E-03	5.0963E-18
100	5.6141E-02	2.9079E-02	1.3739E-02	5.6483E-03	1.7114E-03	4.1703E-18
120	5.1291E-02	2.6012E-02	1.2047E-02	4.8604E-03	1.4464E-03	4.7434E-18
140	4.7514E-02	2.3675E-02	1.0784E-02	4.2826E-03	1.2552E-03	1.9682E-18
160	4.4465E-02	2.1824E-02	9.7994E-03	3.8392E-03	1.1106E-03	1.1778E-18
180	4.1936E-02	2.0313E-02	9.0075E-03	3.4870E-03	9.9708E-04	1.3570E-18
200	3.9795E-02	1.9051E-02	8.3545E-03	3.2000E-03	9.0558E-04	1.4773E-18

Table 4. Concentration of the species B under varying values of α with $k = 1\ s^{-1}$

t	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
0	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
20	8.7679E-01	9.2162E-01	9.5480E-01	9.7762E-01	9.9196E-01	1.0000E+00
40	9.1187E-01	9.4893E-01	9.7316E-01	9.8782E-01	9.9595E-01	1.0000E+00
60	9.2776E-01	9.6022E-01	9.8009E-01	9.9136E-01	9.9725E-01	1.0000E+00
80	9.3731E-01	9.6666E-01	9.8385E-01	9.9320E-01	9.9790E-01	1.0000E+00
100	9.4386E-01	9.7092E-01	9.8626E-01	9.9435E-01	9.9829E-01	1.0000E+00
120	9.4871E-01	9.7399E-01	9.8795E-01	9.9514E-01	9.9855E-01	1.0000E+00
140	9.5249E-01	9.7632E-01	9.8922E-01	9.9572E-01	9.9874E-01	1.0000E+00
160	9.5553E-01	9.7818E-01	9.9020E-01	9.9616E-01	9.9889E-01	1.0000E+00
180	9.5806E-01	9.7969E-01	9.9099E-01	9.9651E-01	9.9900E-01	1.0000E+00
200	9.6020E-01	9.8095E-01	9.9165E-01	9.9680E-01	9.9909E-01	1.0000E+00

From the obtained data shown in Tables 3 and 4, it is once again observed that the fractional-order parameter α gives us an infinite number of degrees of freedom to know about the behavior of concentration of the species A and B. The value of α when taken within the interval $]0, 0.9]$ does not cause the concentration to be completely emptied, but when $\alpha > 0.9$, we observe a rapid decrease in the concentration.

In this way, the effects taking place within the chemical reaction during process of production are obtained with the help of fractional-order non-local operators. Thus, memory effects associated with the governing equations in the chemical reaction (28) are well captured.

5. Conclusion

This research work is about using a recently devised integral transform called the Mohand integral transform for solving fractional-order initial value problems under the Caputo type fractional-order derivative operator. In the existing literature, this transform has so far been used to solve only integer-order differential equations whereas, in the present work, we use it for some practically significant fractional--order mathematical models mostly related to kinetic reactions from the chemical engineering field.

With the help of a theorem proposed and thoroughly derived in the present research work, the fractional-order models under investigation were made possible to be solved exactly, and their required exact solutions were obtained. Fractional--order models revealed a Non-Markovian nature for $\alpha \in (0,1)$ and follow the Markovian process when $\alpha = 1$. This type of hereditary nature of non-local operators revealed some interesting insights about the behavior of the models under investigation, and we could conclude that kinetic rate equations are heavily dependent upon the fractional-order parameter α to produce meaningful results.

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