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UNIFORMLY CONVERGENT HIGHER-ORDER FINITE DIFFERENCE SCHEME FOR SINGULARLY PERTURBED PARABOLIC PROBLEMS WITH NON-SMOOTH DATA

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Abstract. A uniformly convergent higher-order finite difference scheme is constructed and analyzed for solving singularly perturbed parabolic problems with non-smooth data. This scheme involves an average non-standard finite difference with the Richardson extrapolation method for space variables and second-order finite difference approximation for time direction on uniform meshes. The scheme is shown to be second-order convergent in both temporal and spatial directions. Further, the scheme is proven to be uniformly convergent and also confirmed by numerical experiments. Wide numerical experiments are conducted to support the theoretical results and to demonstrate its accuracy. Concisely, the present scheme is stable, convergent, and more accurate than existing methods in the literature.

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1. Introduction

In this paper, we consider a class of singularly perturbed parabolic problems with non-smooth data whose solutions exhibit strong interior layers due to the discontinuous convection coefficient. These types of problems arise in several branches of engineering and applied mathematics, including convection dominated flows in fluid dynamics, heat, and mass transfer in chemical and nuclear engineering. Singularly perturbed parabolic problems are branded by the occurrence of a small parameter that multiplies the highest order derivative, and there exists a boundary or interior layer where the solutions change rapidly. Solutions of singularly perturbed parabolic problems of convection-diffusion typically contain boundary layers [1-5]. However, interior layers occur if coefficient or source functions or the boundary and/or initial conditions are not sufficiently smooth [6] and [7]. When the perturbation parameter is small, very complex phenomena can happen near the point of discontinuity, whose theoretical analysis is not yet well understood. Solving such problems is the most problematic and unresolved problems of fluid mechanics, specifically the behavior of viscous fluids at small viscosity.

Thus, several methods have been established by various authors for different kinds of singularly perturbed parabolic problems with smooth data [8-12]. But, works on problems with non-smooth data are rare. Recently, Chandru et al., [1], proposed a numerical treatment of two-parameter singularly perturbed parabolic convection-diffusion problems with non-smooth data. The optimal error estimate of the upwind scheme on Shishkin-type meshes [10] and an ε-Uniform error estimate of the hybrid numerical scheme [11] for singularly perturbed parabolic problems with interior layers are proposed by Mukherjee and Natesan. These methods are based on piecewise-uniform Shishkin meshes, and most of them are first-order spatial accurate. Therefore, it is necessary to provide a uniformly convergent numerical method with a better accuracy. To achieve this purpose, here we develop a uniformly convergent numerical scheme for solving singularly perturbed parabolic problems with non-smooth data of the form:

$$\begin{cases} L_\varepsilon u \equiv \left(\varepsilon \dfrac{\partial^2 u}{\partial x^2} + a(x)\dfrac{\partial u}{\partial x} - b(x)u - \dfrac{\partial u}{\partial t} \right)(x,t) = f(x,t),\ (x,t) \in D \\ u(x,0) = s(x), \quad x \in \bar{\Omega} \\ u(0,t) = q_0(t), \quad u(1,t) = q_1(t) \end{cases} \tag{1}$$

with perturbation parameter ε, satisfies $0 < \varepsilon << 1$. The solution domain by $D = \Omega \times (0,T]$, $\Omega = (0,1)$, $\Omega^- = (0,d)$, $\Omega^+ = (d,1)$, $d \in (0,1)$, $D^- = \Omega^- \times (0,T]$, $D^+ = \Omega^+ \times (0,T]$, T is positive constant. The convection coefficient $a(x)$ and the source term $f(x,t)$ are sufficiently smooth on $D^- \cup D^+$. Also, the coefficient of the reaction term $b(x)$ is sufficiently smooth on $\bar{D}$ such that

$$\begin{cases} b(x) \geq \beta \geq 0, \quad \text{on } \bar{D} \\ \|[a]\| \leq C,\ \|[f]\| \leq C, \quad \text{at } x = d. \end{cases} \tag{2}$$

and the solution $u(x,t)$ satisfies the following interface conditions

$$[u] = 0, \quad \left[\frac{\partial u}{\partial x} \right] = 0, \quad \text{at } x = d. \tag{3}$$

Let us define the jump of u, denoted by $[u]$, across the point of discontinuity $x = d$ by $[u](d,t) = u(d^+,t) - u(d^-,t)$, wherever $u(d^\pm,t) = \lim_{x \to d^\pm} u(x,t)$. Due to the presence of discontinuity in the convection coefficient $a(x)$, solution $u(x,t)$ to Eq. (1) possesses interior layers of width $O(\varepsilon)$ in the neighborhood of the point $x = d$.

The nature of the interior layer depends on the sign of the convection coefficient $a(x)$ on either side of the line of discontinuity [13-18]. Thus, to stress the existence of strong interior layers, consider the following particular condition

$$\begin{cases} -\alpha_1^* < a(x) < -\alpha_1 < 0, & x < d \\ \alpha_2^* > a(x) > \alpha_2 > 0, & x > d \end{cases} \tag{4}$$

Assume that, the coefficient of convection term provided in Eq. (4) is independent of time t, and then restrict the discontinuities in the spatial variable x only. Moreover, the data $s(x)$, $q_0(t)$ and $q_1(t)$ are assumed to be sufficiently smooth on D and satisfy the compatibility conditions at the two corner points $(0,0)$ and $(1,0)$ with $s(0) = q_0(0)$ and $s(1) = q_1(0)$ and

$$\begin{cases} \varepsilon \dfrac{\partial^2 s(0)}{\partial x^2} + a(0)\dfrac{\partial s(0)}{\partial x} - b(0)s(0) - f(0,0) = \dfrac{\partial q_0(0)}{\partial t}, \\ \varepsilon \dfrac{\partial^2 s(1)}{\partial x^2} + a(1)\dfrac{\partial s(1)}{\partial x} - b(1)s(1) - f(1,0) = \dfrac{\partial q_1(0)}{\partial t}. \end{cases} \tag{5}$$

The compatibility conditions at the transition corner point $(d,0)$ trails similarly, and under these hypotheses the problem in Eqs. (1)-(4) admits a unique solution that one can refer in the book [18].

2. Formulation of the numerical scheme

In this section, a numerical scheme will be described by discretizing the time variable on uniform mesh and using the non-standard methodology of Mickens [7, 8] for the space variable. To discretize the time variable with uniform step size k, $[0,T]$ is partitioned $0 = t_0 < t_1 < ... < t_N = T$ for $t_n = nk$, $k = \dfrac{T}{N}$. Now, at the point $(x, t_{n+\frac{1}{2}})$, the operator in Eq. (1) can be written:

$$\left(\varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} - b(x)u - \frac{\partial u}{\partial t} \right)(x, t_{n+\frac{1}{2}}) = f(x, t_{n+\frac{1}{2}}) \tag{6}$$

Taylor's series expansion to $u(x,t_n)$, $u(x,t_{n+1})$ about the point, $(x, t_{n+\frac{1}{2}})$, gives

$$u_t(x, t_{n+\frac{1}{2}}) = \frac{u(x,t_{n+1}) - u(x,t_n)}{k} + \tau_1, \tag{7}$$

where $\tau_1 = -\dfrac{k^2}{24} u_{ttt}(x, t_{n+0.5})$. This indicates that the error estimate of time discretization is bounded and given by

$$\|E_n\|_\infty \le Ck^2, \tag{8}$$

where $C = \dfrac{1}{24}\left\| u_{ttt}(x, t_{n+\frac{1}{2}}) \right\|$, $\forall n = 1, 2, \ldots N$, is a constant independent of ε and k. Also, let's take the average of all terms of Eq. (6) except for term containing a derivative concerning time, which is written as

$$\left(\varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} - b(x)u - f \right)(x, t_{n+\frac{1}{2}}) = \frac{1}{2} L^*_{x,f}\left(u(x, t_{n+1}) + u(x, t_n) \right) \tag{9}$$

where, $L^*_{x,f} u(x, t_n) = L^*_x u(x, t_n) - f(x, t_n)$,

$$L^*_x u(x, t_n) = \left(\varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} - b(x)u \right)(x, t_n).$$

Substituting both Eq. (7) and (9) into Eq. (6), we get the problem

$$\begin{cases} \left(L^*_x - \dfrac{2}{k} \right) u(x, t_{n+1}) + \left(L^*_x + \dfrac{2}{k} \right) u(x, t_n) = f(x, t_{n+1}) + f(x, t_n) \\ u(0, t_{n+1}) = q_0(t_{n+1}), \quad u(1, t_{n+1}) = q_1(t_{n+1}), \quad u(x, 0) = s(x), \ x \in \bar{\Omega} \end{cases} \tag{10}$$

This gives the semi-discretize approximation $u(x, t_{n+1})$ of Eq. (10) to the exact solution $u(x,t)$ of Eq. (1) at the time levels $t_{n+1} = (n+1)k$. To discretize the space variable, assume that $\bar{\Omega}^M$ denotes the partition of $[0,1]$ into M subintervals such that $0 = x_0 < x_1 < \ldots < x_{\frac{M}{2}} = d < x_{\frac{M}{2}+1} \ldots < x_M = 1$, $x_m = mh$, $h = \dfrac{1}{M}$, and then the tensor-product grids $\bar{D}^{M,N}$. Undertake the notation $U^n_m \approx u\left(x_m, t_n\right)$, and the discrete problem to Eq. (10) is given by

$$\begin{cases} L^{M,N}_\varepsilon U^{n+1}_m \equiv \left(L^*_M - \dfrac{2}{k} \right) U^{n+1}_m + \left(L^*_M + \dfrac{2}{k} \right) U^n_m = f^{n+1}_m + f^n_m, \\ U(x_m, 0) = s(x_m), \ x_m \in \bar{\Omega}^M, \\ U(0, t_{n+1}) = q_0(t_{n+1}), \ u(1, t_{n+1}) = q_1(t_{n+1}), \ t_{n+1} \in \left[0, T\right]^N, \end{cases} \tag{11}$$

where

$$\begin{cases} L_M^* U_m^{n+1} = \varepsilon \dfrac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{\phi_m^2} + a_m \dfrac{U_{m+1}^{n+1} - U_m^{n+1}}{h} - b_m U_m^{n+1} + \tau_2 \\ L_M^* U_m^n = \varepsilon \dfrac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{\phi_m^2} + a_m \dfrac{U_{m+1}^n - U_m^n}{h} - b_m U_m^{n+1} + \tau_3 \end{cases} \tag{12}$$

for the truncation terms in space direction given by

$$\tau_2 = -\frac{h}{2}\left(U_{xx}\right)_m^{n+1} + O(h^2) \quad \text{and} \quad \tau_3 = -\frac{h}{2}\left(U_{xx}\right)_m^n + O(h^2) \tag{13}$$

Within the nonstandard finite difference methodology of Mickens [7, 8], the denominator

$$\phi_m^2\left(\varepsilon, h\right) = \frac{h\varepsilon}{a_m}\left(\exp\left(\frac{ha_m}{\varepsilon}\right) - 1\right) \tag{14}$$

Now, incorporating the initial and boundary conditions given, the scheme becomes

$$\begin{cases} U_m^0 = s(x_m), \quad \text{for } m = 0,1,2,\ldots,M \\ L_\varepsilon^{M,N} U_m^{n+1} = H_m^{n+1}, \quad \text{for } m = 1,2,\ldots,M-1 \\ U_0^{n+1} = q_0(t_{n+1}), \quad U_M^{n+1} = q_1(t_{n+1}), \ \forall\, n = 0,1,\ldots,N. \end{cases} \tag{15}$$

where

$$L_\varepsilon^{M,N} U_m^{n+1} = \overset{-}{A}_m U_{m-1}^{n+1} + \overset{0}{A}_m U_m^{n+1} + \overset{+}{A}_m U_{m+1}^{n+1} + [\overset{-}{B}_m U_{m-1}^n + \overset{0}{B}_m U_m^n + \overset{+}{B}_m U_{m+1}^n],$$

and $H_m^{n+1} = f_m^{n+1} + f_m^n$, $m = 1,2,\ldots, M-1$ and $n = 0,1,\ldots, N$.

Moreover, the coefficients are

$$\overset{-}{A}_m = \overset{-}{B}_m = \frac{\varepsilon}{\phi_m^2}, \quad \overset{+}{A}_m = \overset{+}{B}_m = \frac{\varepsilon}{\phi_m^2} + \frac{a_m}{h},$$

$$\overset{0}{A}_m = -\left(\frac{2\varepsilon}{\phi_m^2} + \frac{a_m}{h} + b_m + \frac{2}{k}\right), \quad \text{and} \quad \overset{0}{B}_m = -\left(\frac{2\varepsilon}{\phi_m^2} + \frac{a_m}{h} + b_m - \frac{2}{k}\right)$$

3. Consistency and stability of the scheme

Local truncation errors refer to the differences between the original differential equation and its finite difference approximations at grid points. To investigate the consistency of the method, we have the norm of local truncation errors $\left|T_m^{n+1}\right|$ obtained in both Eqs. (13) ad (8) for the space and time directions respectively given by

$$\left|T_m^{n+1}\right| = \left|\tau_2 + \tau_3 + \tau_1\right| \le C_1 h + Ck^2. \tag{16}$$

where the constant are $C_1 = \left|-\frac{1}{2}\left(U_{xx}\right)_m^{n+1}\right| + \left|-\frac{1}{2}\left(U_{xx}\right)_m^{n}\right|$ and $C = \frac{1}{24}\left|u_{ttt}(x, t_{n+\frac{1}{2}})\right|$.

Thus, the right-hand side hand of Eq. (16) vanishes as $h \to 0$ and $k \to 0$ implied that $T_m^{n+1} \to 0$. Hence, the scheme is consistent with the order of convergences $O\left(h + k^2\right)$.

The Von Neumann stability technique is applied to investigate the stability of the developed scheme in Eq. (15), by assuming that the solution of Eq. (15) at the grid point $\left(x_m, t_n\right)$ is given by

$$U_m^n = \xi^n e^{i\theta m} \tag{17}$$

where $i = \sqrt{-1}$, θ is the real number and ξ denotes the amplitude factor. Then, substituting Eq. (17) into the homogeneous part of Eq. (15) leads to:

$$\xi = \frac{-\left[B_m^- e^{-1} + B_m^0 + B_m^+ e^1\right]}{A_m^- e^{-1} + A_m^0 + A_m^+ e^1} \tag{18}$$

Substituting the values of coefficients considered under Eq. (15) into Eq. (18) with stability conditio $|\xi| \le 1$ is satisfied. Because

$$|\xi| = \left| -\frac{\frac{\varepsilon}{\phi_m^2} e^{-1} - \frac{2\varepsilon}{\phi_m^2} - \frac{a_m}{h} - b_m + \frac{2}{k} + \left(\frac{\varepsilon}{\phi_m^2} + \frac{a_m}{h}\right) e^1}{\frac{\varepsilon}{\phi_m^2} e^{-1} - \frac{2\varepsilon}{\phi_m^2} - \frac{a_m}{h} - b_m - \frac{2}{k} + \left(\frac{\varepsilon}{\phi_m^2} + \frac{a_m}{h}\right) e^1} \right| \le 1.$$

This leads to:

$$\left|\frac{\varepsilon}{\phi_m^2} e^{-1} - \frac{2\varepsilon}{\phi_m^2} - \frac{a_m}{h} - b_m + \frac{2}{k} + \left(\frac{\varepsilon}{\phi_m^2} + \frac{a_m}{h}\right) e^1\right| \le \left|\frac{\varepsilon}{\phi_m^2} e^{-1} - \frac{2\varepsilon}{\phi_m^2} - \frac{a_m}{h} - b_m - \frac{2}{k} + \left(\frac{\varepsilon}{\phi_m^2} + \frac{a_m}{h}\right) e^1\right|,$$

$$\Rightarrow \left|\frac{2}{k}\right| \le \left|-\frac{2}{k}\right|.$$

Hence, $|\xi| \le 1$, which implies the developed finite difference scheme in Eq. (15) is unconditionally stable. Therefore, the proposed finite difference scheme is consistent and stable which guarantees it is convergent by Lax's equivalence theorem [19].

4. Richardson extrapolation

This technique is a convergence acceleration technique that involves a combination of two computed approximations of a solution. The combination goes out to be an improved approximation [12, 19]. In this work, we apply the Richardson extrapolation method on the space variable only. So that from Eq. (16), we have

$$\left| u(x_m, t_{n+1}) - U_m^{n+1} \right| \le Ch \tag{19}$$

where $u(x_m, t_{n+1})$ and U_m^{n+1} are exact and approximate solutions respectively, C is constant free from ε and h. Let D_{2M}^N be the mesh found by dividing each mesh interval in D_M^N and symbolize the calculation of the solution on D_{2M}^N by $\bar{U}_m^{n+1}$. Consider Eq. (19) works for any $h \ne 0$, which implies:

$$u(x_m, t_{n+1}) - U_m^{n+1} \le Ch + R_M^N, \quad (x_m, t_{n+1}) \in D_M^N. \tag{20}$$

Also, consider Eq. (19) works for any $\frac{h}{2} \ne 0$, that leads to:

$$u(x_m, t_{n+1}) - \bar{U}_m^{n+1} \le C\frac{h}{2} + R_{2M}^N, \quad (x_m, t_{n+1}) \in D_{2M}^N. \tag{21}$$

where the remainders R_M^N and R_{2M}^N are $O(h^2)$. A combination of inequalities in Eqs. (20) and (21) leads to $u(x_m, t_{n+1}) - \left(2\bar{U}_m^{n+1} - U_m^{n+1}\right) \approx O(h^2)$ which proposes:

$$\left(U_m^{n+1}\right)^{ext} = 2\bar{U}_m^{n+1} - U_m^{n+1} \tag{22}$$

is also an approximate solution of $u(x_m, t_{n+1})$. The solution obtained by Eq. (22) approximates the solution with an estimated truncation error

$$\left| u(x_m, t_{n+1}) - \left(U_m^{n+1}\right)^{ext} \right| \le C\,h^2. \tag{23}$$

Thus, from Eq. (23) for the order in the spatial direction with Eqs. (7) and (8) for the order of temporal direction, we can conclude that for $\left(U_m^{n+1}\right)^{ext}$ be the solution of Eq. (15) and u_m^{n+1} be the solution to Eq. (1) at the grid point (x_m, t_{n+1}):

$$\max_{0\le m\le M}\left|u_m^{n+1}-\left(U_m^{n+1}\right)^{ext}\right|\le C\left(h^2+k^2\right),\ n=0,1,\ldots,\ N, \tag{24}$$

where a constant C is independent of ε, h, and k.

5. Numerical illustrations

To demonstrate the applicability of the developed scheme computationally, we consider three examples. The exact solution for such types of problems is not available, so that maximum absolute errors at all the mesh points are evaluated before and after the Richardson extrapolation using the formula

$$E_\varepsilon^{M,N}=\max_{0\le m\le M;\,0\le n\le N}\left|U_m^{n+1}-U_{2m}^{n+1}\right| \quad\text{and}\quad ER_\varepsilon^{M,N}=\max_{0\le m\le M;\,0\le n\le N}\left|\left(U_m^{n+1}\right)^{ext}-\left(U_{2m}^{n+1}\right)^{ext}\right|$$

respectively, where U_m^{n+1} is an approximate solution obtained using a constant space mash size h and time step k, and U_{2m}^{n+1} is also an approximate solution produced using space step size $\frac{h}{2}$.

Example 1. Consider the singularly perturbed parabolic problem

$$\begin{cases}\varepsilon u_{xx}+a(x)u_x-x(1-x)u-u_t=f(x,t),\ (x,t)\in(0,1)\times(0,1]\\ u(x,0)=0,\quad 0\le x\le 1\\ u(0,t)=t^2,\quad u(1,t)=0,\quad 0<t\le 1\end{cases}$$

$$a(x)=\begin{cases}-(1+x(1+x)), & 0\le x\le 0.5\\ (1+x(1+x)), & 0.5<x\le 1\end{cases}\quad\text{and}\quad f(x,t)=\begin{cases}2(1+x^2)t^2, & 0\le x\le 0.5\\ 3(1+x^2)t^2, & 0.5<x\le 1\end{cases}$$

A comparison of maximum absolute errors for Example 1 is given in Table 1 with solution behavior in Figure 1a.

Example 2. Consider the parabolic problem

$$\begin{cases}\varepsilon u_{xx}+a(x)u_x-u_t=f(x,t),\ (x,t)\in(0,1)\times(0,1]\\ u(x,0)=0,\quad 0\le x\le 1\\ u(0,t)=0=u(1,t),\quad 0<t\le 1\end{cases}$$

$$a(x)=\begin{cases}-(2+x^2), & 0\le x\le 0.5\\ 3-x^2, & 0.5<x\le 1\end{cases}\quad\text{and}\quad f(x,t)=\begin{cases}2x\exp(-t)t^2, & 0\le x\le 0.5\\ 2(1-x)\exp(-t)t^2, & 0.5<x\le 1\end{cases}$$

The computed maximum absolute errors and the corresponding order of convergence for Example 2 are presented in Tables 2 and 3 in the cases of before and after Richardson extrapolation. Also, the solution behavior in Figure 1b and log-log plot in Figure 2a are given.

Example 3. Consider the problem

$$\begin{cases} \varepsilon u_{xx} + a(x)u_x - u_t = f(x,t), \ (x,t) \in (0,1) \times (0,1] \\ u(x,0) = 0, \quad 0 \le x \le 1 \\ u(0,t) = t^2, \quad u(1,t) = 0, \quad 0 < t \le 1 \end{cases}$$

where $a(x) = \begin{cases} -1, & 0 \le x \le 0.5 \\ 1, & 0.5 < x \le 1 \end{cases}$ and $f(x,t) = \begin{cases} 2xt, & 0 \le x \le 0.5 \\ 2(1-x)t, & 0.5 < x \le 1 \end{cases}$.

A comparison of maximum absolute errors for Example 3 is given in Table 4 with a log-log plot in Figure 2b.

Table 1. Comparison of maximum absolute errors for Example 1

$\varepsilon \downarrow M = N \rightarrow$	32	64	128	256
Present Method				
10^{-1}	1.3124e-03	2.1065e-04	5.3718e-05	1.3570e-05
10^{-4}	8.1406e-04	3.3685e-04	1.4815e-04	6.8379e-05
10^{-6}	8.1406e-04	3.3685e-04	1.4815e-04	6.8379e-05
10^{-8}	8.1406e-04	3.3685e-04	1.4815e-04	6.8379e-05
10^{-10}	8.1406e-04	3.3685e-04	1.4815e-04	6.8379e-05
Results in [15]				
10^{-1}	4.5469e-03	2.1384e-03	1.0448e-03	5.1766e-04
10^{-4}	7.3215e-02	2.7267e-02	8.3339e-03	2.7829e-03
10^{-6}	7.3246e-02	2.7273e-02	8.3349e-03	2.7828e-03
10^{-8}	7.3246e-02	2.7273e-02	8.3349e-03	2.7828e-03
10^{-10}	7.3246e-02	2.7273e-02	8.3349e-03	2.7827e-03

Table 2. Maximum absolute errors for Example 2 when $M = N$

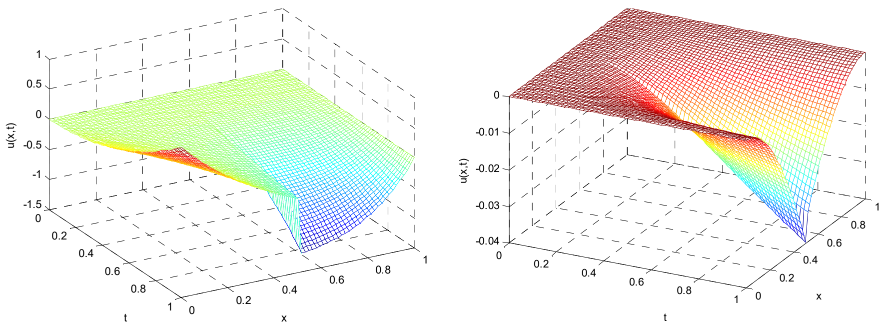
$\varepsilon \downarrow N$	32	64	128	256	512
After Extrapolation					
10^{-1}	2.3784e-04	6.8070e-05	1.7754e-05	4.5055e-06	1.1331e-06
10^{-4}	1.3531e-05	3.4015e-06	8.5267e-07	2.1345e-07	5.5331e-08
10^{-6}	1.3531e-05	3.4015e-06	8.5267e-07	2.1345e-07	5.5331e-08
10^{-8}	1.3531e-05	3.4015e-06	8.5267e-07	2.1345e-07	5.5331e-08
10^{-10}	1.3531e-05	3.4015e-06	8.5267e-07	2.1345e-07	5.5331e-08
Before Extrapolation					
10^{-1}	1.0164e-03	3.8963e-04	1.6087e-04	7.1577e-05	3.3541e-05
10^{-4}	9.0542e-04	4.5979e-04	2.3167e-04	1.1628e-04	5.8250e-05
10^{-6}	9.0542e-04	4.5979e-04	2.3167e-04	1.1628e-04	5.8250e-05
10^{-8}	9.0542e-04	4.5979e-04	2.3167e-04	1.1628e-04	5.8250e-05
10^{-10}	9.0542e-04	4.5979e-04	2.3167e-04	1.1628e-04	5.8250e-05

Table 3. Rate of convergence for Example 2, when $M = N$

$\varepsilon \downarrow N$	32	64	128	256
After Extrapolation				
10^{-1}	1.8049	1.9389	1.9784	1.9914
10^{-4}	1.9920	1.9961	1.9981	1.9477
10^{-6}	1.9920	1.9961	1.9981	1.9477
10^{-8}	1.9920	1.9961	1.9981	1.9477
10^{-10}	1.9920	1.9961	1.9981	1.9477
Before Extrapolation				
10^{-1}	1.3833	1.2762	1.1683	1.0936
10^{-4}	0.9776	0.9889	0.9945	0.9973
10^{-6}	0.9776	0.9889	0.9945	0.9973
10^{-8}	0.9776	0.9889	0.9945	0.9973
10^{-10}	0.9776	0.9889	0.9945	0.9973

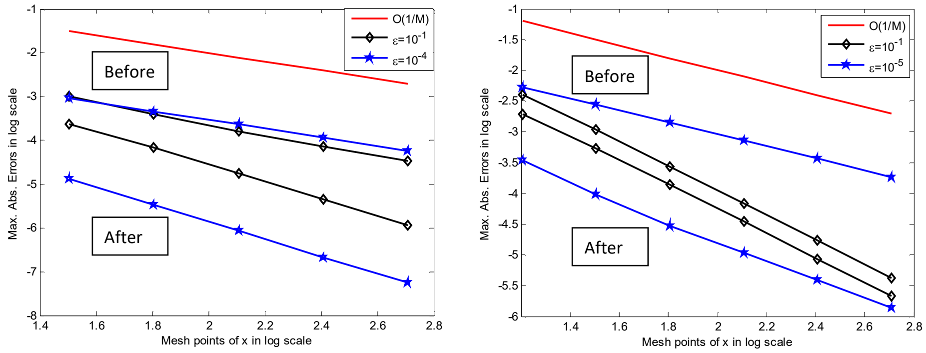
Table 4. Comparison of maximum absolute errors for Example 3 when $M = N$

$\varepsilon \downarrow N$	32	64	128	256	512
After Extrapolation					
10^{-1}	5.3343e-04	1.3647e-04	3.4315e-05	8.5912e-06	2.1486e-06
10^{-5}	9.4849e-05	2.9819e-05	1.0681e-05	3.8488e-06	1.3861e-06
10^{-10}	9.4849e-05	2.9819e-05	1.0681e-05	3.8488e-06	1.3861e-06
Before Extrapolation					
10^{-1}	1.0686e-03	2.7073e-04	6.7910e-05	1.6992e-05	4.2488e-06
10^{-5}	2.7667e-03	1.4248e-03	7.2263e-04	3.6385e-04	1.8256e-04
10^{-10}	2.7667e-03	1.4248e-03	7.2263e-04	3.6385e-04	1.8256e-04
Results in [14]					
10^{-1}	4.6571e-3	1.9099e-3	8.8214e-4	4.2631e-4	2.0976e-4
10^{-5}	9.9736e-3	4.1094e-3	1.7641e-3	8.2587e-4	3.9692e-4
10^{-10}	9.9737e-3	4.1095e-3	1.7641e-3	8.2580e-4	3.9691e-4



(a) Example 1 (b) Example 2

Fig. 1. Numerical solution when $\varepsilon = 10^{-4}$, $M = N = 64$ for Examples 1 and 2



(a) Example 1 (b) Example 3

Fig. 2. Log-log plot of maximum absolute errors before and after extrapolation at the number of intervals for Examples 2 and 3

6. Conclusion

In this paper, we have developed and analyzed a uniformly convergent higher--order finite difference scheme for solving singularly perturbed parabolic problems with non-smooth data. To develop this scheme, we use a uniform mesh for the temporal direction and followed by a nonstandard finite difference methodology of Mickens in the spatial direction. We have recognized the maximum principle and stability results for continuous and discrete problems, and their decomposition in smooth and layer components. It has been shown theoretically that the developed scheme is uniformly convergent second-order accuracy and confirmed with experimental results in Table 3. Also, the accuracy of computational verification tested and compared as displayed in Tables 1, 2, and 4 for three considered model examples. Moreover, we also observed that the proposed scheme has a better numerical accuracy compared to the existing methods in the literature. To verify the interior layer that happens due to a discontinuous convection coefficient and source function. This layer behavior is simulated in Figure 1. From Figure 2, we illustrate the contribution of the Richardson extrapolation technique gives a more accurate solution with a high rate of convergence.

Generally, the proposed method is uniformly convergent of order two (higher--order) for solving singularly perturbed parabolic problems with non-smooth data. Furthermore, the method is stable, convergent, and gives a more accurate solution than the existing methods in the literature.

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