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ASYMPTOTIC ANALYSIS OF A CLOSED G-NETWORK OF UNRELIABLE NODES

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Abstract. A closed exponential queueing G-network of unreliable multi-server nodes was studied under the asymptotic assumption of a large number of customers. The process of changing the number of functional servers in network nodes was considered as the birth--death process. The process of changing the number of customers at the nodes was considered as a continuous-state Markov process. It was proved that its probability density function satisfies the Fokker-Planck-Kolmogorov equation. The system of differential equations for the first-order and second-order moments of this process was derived. This allows us to predict the expectation, the variance and the pairwise correlation of the number of customers in the G-network nodes both in the transient and steady state.

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1. Introduction

G-networks are generalized queueing networks that are primarily different from Jackson and BCMP networks in that they are networks of queues with several types of customers: positive customers, negative customers and in some cases triggers. Negative customers and triggers are not served. When a negative customer arrives at a queueing system (queue, node), one or a group of positive customers is removed (or "killed") in a non-empty queue, while the queued trigger displaces customers and moves positive customers from one node to some other node. G-networks were first introduced by E. Gelenbe and have been studied in a steady state since the 90s [1-3]. G-networks are of great interest for extending the multiplicative theory of queueing networks. Their field of application is modelling computing systems and networks, evaluating their performance, modelling biophysical neural networks, pattern recognition tasks and others [4-6]. At the present time, many scientific works are dedicated to the study of G-networks. G-networks in the transient state were studied by M. Matalytski [7, 8].

It is important to make mention of random neural networks (RNNs). RNNs are a class of neural networks (NNs) that can also be seen as a specific type of queueing networks. RNNs are closely linked to the G-networks. The RNN model was introduced by E. Gelenbe in 1989 [9]. The design of the model was inspired from the biological behavior of neuron circuits in the neo-cortex. Their areas of application are machine learning problems, optimization, image processing, associative memories, etc. [10-12].

The purpose of this paper is to asymptotically study a Markov G-network of unreliable queueing nodes. The unreliability of nodes lies in the fact that their servers can be broken down and be repaired according to a certain statistical law. The study of queueing networks with multi-server nodes in case of server breakdowns and repairs is important for practical applications. It is a priori obvious that the queue lengths at the nodes depend on systematic server failures. Asymptotic analysis implies an approximation method of queueing network study under the assumption of a large but limited number of customers (requests) [13-15].

Discrete (discontinuous-state) Markov processes are usually used to determine the state of queueing networks. In this paper, the passage to the limit from a Markov chain to a continuous-state Markov process was considered. In contrast to discontinuous processes, continuous processes in any small time interval $\Delta t \to 0$ have some small change in the state $\Delta x \to 0$. The mathematical approach used in this paper is based on a discrete model of a continuous Markov process described in many books on the theory of diffusion Markov processes [16].

2. Formulation of the problem

A closed exponential G-network including nodes S_i, $i=\overline{1,n}$, and zero node S_0 is under study. Suppose that the network is designed to serve a finite and, moreover, a constant number of customers K. Node S_0 is interpreted as a dependent source of customers, generating customers in the G-network only at the moment customers arrive to S_0. We assume that the arrival of customers from S_0 forms Poisson process of rate $\lambda = \lambda_0 k_0(t)$, the rate parameter is proportional to the number of customers in the source $k_0(t)$. The flow of arriving customers is divided into two types: positive and negative. Thus, the probability of a positive customer arriving from source S_0 to node S_i in time interval $[t,\ t+\Delta t]$ is $\lambda_0 p_{0i}^{+}\Delta t + o(\Delta t)$, negative – $\lambda_0 p_{0i}^{-}\Delta t + o(\Delta t)$, $i=\overline{1,n}$, $\sum_{i=1}^{n}(p_{0i}^{+}+p_{0i}^{-})=1$.

Each of the nodes S_i is an unreliable queueing system with an infinite queue length, the number of identical servers is m_i, $i=\overline{1,n}$. Servers in nodes S_i, $i=\overline{1,n}$, are subject to random failure. The continuous functional time of each server in S_i is exponentially distributed with the rate parameter α_i, $i=\overline{1,n}$. The server lifespan does not depend on whether the device is busy or not. The server immediately starts to be repaired after the failure. The server repair time in S_i also has an exponential distribution with the rate parameter β_i, $i=\overline{1,n}$. Suppose if the server fails during the customer service time, the interrupted customer will be completed after the repair of the server. Let us assume that the server service time, server uptime, and server repair time are independent random variables. Node S_0 is a queueing system without a waiting area and it has K identical reliable servers.

All servers of the node S_i are identical, and they have exponentially distributed service time for positive customers. Let μ_i is the reciprocal of their mean service time, $i=\overline{1,n}$. Positive customers are served in order of arrival. A negative customer arriving at some non-empty network node removes one customer in this node and immediately transfers to source S_0 without receiving any service at the node. Therefore, only positive customers can be served at each node, and so they are usually referred to simply as customers for brevity. If the service time of positive customers has an exponential distribution, then there is no difference in terms of which customer is removed from the node.

Customer routing is defined as follows. A positive customer served at node S_i with probability p_{ij}^{+} transfers to node S_j as a positive customer, or with probability p_{ij}^{-} transfers as a negative customer, or with probability $p_{i0}=1-\sum_{j=1}^{n}\left(p_{ij}^{+}+p_{ij}^{-}\right)$ leaves the network and joins source S_0, $i,j=\overline{1,n}$.

The state of the network under study at time t is represented by a vector

$$(z(t);\ k(t))=(z_1(t),\ z_2(t),...,\ z_n(t);\ k_1(t),\ k_2(t),...,\ k_n(t)), \tag{1}$$

where $z_i(t)$ is the number of functional servers, $k_i(t)$ is the number of customers in the queueing system S_i at the time t, $0\le z_i(t)\le m_i$, $0\le k_i(t)\le K$, $i=\overline{1,n}$, $t\in[0,+\infty)$. Since the network is closed, it is obvious that $k_0(t)=K-\sum_{i=1}^{n}k_i(t)$. The state (1) can be viewed as two simultaneous random processes $z(t)$ and $k(t)$.

The purpose of the study is to determine states probabilities, expected value and variance of the vector $z(t)$; derive systems of differential equations for the first-order and second-order moments of the $k(t)$ in the asymptotic case of a large K.

3. The process of changing the number of functional servers in network nodes

The process of changing the number of functional servers in network nodes is $z(t)=(z_1(t),\ z_2(t),...,\ z_n(t))$. Server failures and repairs in different queueing nodes are assumed to occur completely independently and regardless of the number of requests in these nodes. Therefore, the vector elements $z_i(t)$ are independent stochastic processes and $z_i(t)$ are not determined by $k_j(t),\ i,j=\overline{1,n}$.

The process $z_i(t)$ can be considered as the birth-death process with birth rates β_i, death rates α_i and the finite integer state space $Z_i=\{0,1,2,...,m_i\},\ i=\overline{1,n}$. Denote $p_{z_i}^{(i)}(t)=P(z_i(t)=z_i)$ is the probability that node S_i has z_i functional servers at time t, $z_i\in\mathbb{Z},\ 0\le z_i\le m_i,\ i=\overline{1,n}$. The differential equations for the probabilities $p_{z_i}^{(i)}(t)$ are well known [16]:

$$\begin{cases} p_0^{(i)\prime}(t)=-\beta_i p_0^{(i)}(t)+\alpha_i p_1^{(i)}(t), \\ p_{z_i}^{(i)\prime}(t)=-(\alpha_i+\beta_i)p_{z_i}^{(i)}(t)+\alpha_i p_{z_i+1}^{(i)}(t)+\beta_i p_{z_i-1}^{(i)}(t),\ 1\le z_i\le m_i-1, \\ p_{m_i}^{(i)\prime}(t)=\beta_i p_{m_i-1}^{(i)}(t)-\alpha_i p_{m_i}^{(i)}(t),\ i=\overline{1,n}. \end{cases} \tag{2}$$

The non-stationary probability distribution $p_{z_i}^{(i)}(t),\ 0\le z_i\le m_i,\ i=\overline{1,n}$, can be found by solving system (2) with a certain initial condition. The expected value and the variance of the number of functional servers at the node S_i are

$$M_{z_i}(t)=M(z_i(t))=\sum_{z_i=1}^{m_i} z_i p_{z_i}^{(i)}(t),\ \sigma_{z_i}^2(t)=D(z_i(t))=\sum_{z_i=1}^{m_i} z_i^{\,2} p_{z_i}^{(i)}(t)-\left(\sum_{z_i=1}^{m_i} z_i p_{z_i}^{(i)}(t)\right)^2.$$

4. The process of changing the number of customers at the nodes

The process $k(t)=(k_1(t),k_2(t),...,k_n(t))$ of changing the number of customers at the nodes related to the customer service process, to customer routing between network nodes and to traffic rerouting or to traffic destruction due to the influence of negative customers is a continuous-time Markov chain with a finite state space. The process $k(t)$ is determined by the service process at nodes and therefore depends on the number of functional servers. This means that $k(t)$ is determined by $z(t)$. To sum up, it may be said $k(t)$ is a nested process with respect to $z(t)$. Probabilities $p_{z_i}^{(i)}(t)$ are assumed to be predetermined from (2).

Theorem 1. In the asymptotic case of a large number of customers $K\to\infty$ the probability density function $p(x,t)$ of the random process $\xi(t)=\dfrac{k(t)}{K}=\left(\dfrac{k_1(t)}{K},\dfrac{k_2(t)}{K},...,\dfrac{k_n(t)}{K}\right)$ provided that it is differentiable with respect to t and twice continuously differentiable with respect to $x_i,\ i=\overline{1,n}$, satisfies up to $O(\varepsilon^2)$, where $\varepsilon=K^{-1}$, the Fokker-Planck-Kolmogorov equation

$$\frac{\partial p(x,t)}{\partial t} = -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(A_i(x,t)p(x,t)\right) + \frac{\varepsilon}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}\left(B_{ij}(x,t)p(x,t)\right), \tag{3}$$

with drifts $A_i(x,t)$ and diffusion coefficients $B_{ij}(x,t)$:

$$A_i(x,t) = \lambda_0\left(1-\sum_{i=1}^{n}x_i\right)(p_{0i}^{+}-p_{0i}^{-}) + \sum_{j=1}^{n}\sum_{z_j=0}^{m_j}\mu_j \min\left(x_j, w_j\right) p_{z_j}^{(j)}(t)(p_{ji}^{+}-p_{ji}^{-}-\delta_{ij}) -$$
$$-\sum_{z_i=0}^{m_i}\mu_i \min\left(x_i, w_i\right) p_{z_i}^{(i)}(t)\sum_{j=1}^{n}p_{ij}^{-}(1-\theta(x_j)), \tag{4}$$

$$B_{ii}(x,t) = \lambda_0\left(1-\sum_{i=1}^{n}x_i\right)(p_{0i}^{+}-p_{0i}^{-}) + \sum_{j=1}^{n}\sum_{z_j=0}^{m_j}\mu_j \min\left(x_j, w_j\right) p_{z_j}^{(j)}(t)(p_{ji}^{+}-p_{ji}^{-}+\delta_{ij}) +$$
$$+\sum_{z_i=0}^{m_i}\mu_i \min\left(x_i, w_i\right) p_{z_i}^{(i)}(t)\sum_{j=1}^{n}p_{ij}^{-}(1-\theta(x_j)), \tag{5}$$

$$B_{ij}(x,t) = \sum_{z_i=0}^{m_i}\mu_i \min\left(x_i, w_i\right) p_{z_i}^{(i)}(t)(p_{ji}^{-}-p_{ji}^{+}),\ i \neq j,$$

δ_{ij} is the Kronecker delta, $\theta(x) = \begin{cases}1, & x > 0,\\ 0, & x \le 0,\end{cases}$ is the Heaviside step function.

Proof. Denote I_i as n-vector with zero components excluding i-th, that is equal to 1. The assumptions made in the formulation of the problem determine that in a short time $\Delta t \to 0$ Markov process $k(t)$ can make one of the following transitions to the state $(k, t+\Delta t)$:

1) from the state $(k - I_i, t)$ with probability

$$\lambda_0 p_{0i}^{+}\left(K - \sum_{i=1}^{n}k_i(t)+1\right)\Delta t + o\left(\Delta t\right),$$

that corresponds to the arrival of a positive customer from S_0 to S_i, $i=\overline{1,n}$;

2) from the state $(k + I_i, t)$ with probability

$$\lambda_0 p_{0i}^{-}\left(K - \sum_{i=1}^{n}k_i(t)+1\right)\Delta t +$$
$$+\sum_{z_i=0}^{m_i}\mu_i \min\left(k_i(t)+1, z_i(t)\right) p_{z_i}^{(i)}(t)\left(p_{i0} + p_{ij}^{-}(1-\theta(k_j(t)))\right)\Delta t + o\left(\Delta t\right),$$

which means the completion of customer service in the node S_i and customer transition to external environment S_0, or the arrival of a negative customer into the node S_i from the source S_0, or the transition of the customer as a negative from S_i to S_j, when S_j does not contain customers, $i, j = \overline{1,n}$;

3) from the state $(k + I_i + I_j, t)$ with probability

$$\sum_{z_i=0}^{m_i} \mu_i \min\left(k_i(t)+1,\ z_i(t)\right) p_{z_i}^{(i)}(t) p_{ij}^{+} \Delta t + o\left(\Delta t\right),$$

which means the customer served in S_i transferred to S_j as positive, $i, j = \overline{1,n}$;

4) from the state $(k + I_i + I_j, t)$ with probability

$$\sum_{z_i=0}^{m_i} \mu_i \min\left(k_i(t)+1,\ z_i(t)\right) p_{z_i}^{(i)}(t) p_{ij}^{-} \Delta t + o\left(\Delta t\right),$$

which means the customer served in S_i transferred to S_j as negative, $i, j = \overline{1,n}$;

5) from the state (k, t) in case of no customers transfer with probability

$$1 - \left(\sum_{i=1}^{n} \lambda_0 p_{0i}^{+} \left(K - \sum_{i=1}^{n} k_i(t) \right) + \sum_{i=1}^{n} \lambda_0 p_{0i}^{-} \left(K - \sum_{i=1}^{n} k_i(t) \right) + \right.$$
$$+ \sum_{i,j=1}^{n} \sum_{z_i=0}^{m_i} \mu_i \min\left(k_i(t), z_i(t)\right) p_{z_i}^{(i)}(t) p_{ij}^{-} (1 - \theta(k_j(t))) +$$
$$\left. + \sum_{i=1}^{n} \sum_{z_i=0}^{m_i} \mu_i \min\left(k_i(t), z_i(t)\right) p_{z_i}^{(i)}(t) \right) \Delta t + o\left(\Delta t\right);$$

6) from other states with probability $o\left(\Delta t\right)$.

In regard to the transitions listed above in the short time Δt, using the law of total probability, the following system of Kolmogorov difference-differential equations is valid for the probability $P(k,\ t) = P(k(t) = k)$:

$$\frac{dP(k,t)}{dt} = \sum_{i=1}^{n} \lambda_0 p_{0i}^{+} \left(K - \sum_{i=1}^{n} k_i(t) + 1 \right) P(k - I_i, t) + \left(\sum_{i=1}^{n} \lambda_0 p_{0i}^{-} \left(K - \sum_{i=1}^{n} k_i(t) + 1 \right) + \right.$$
$$\left. + \sum_{i,j=1}^{n} \sum_{z_i=0}^{m_i} \mu_i \min\left(k_i(t)+1,\ z_i(t)\right) p_{z_i}^{(i)}(t) \left(p_{i0} + p_{ij}^{-} (1 - \theta(k_j(t))) \right) \right) P(k + I_i,\ t) +$$

$$+\sum_{i,j=1}^{n}\sum_{z_i=0}^{m_i}\mu_i \min\big(k_i(t)+1, z_i(t)\big)\, p_{z_i}^{(i)}(t)\, p_{ij}^{+} P(k+I_i-I_j,t)+$$

$$+\sum_{i,j=1}^{n}\sum_{z_i=0}^{m_i}\mu_i \min\big(k_i(t)+1, z_i(t)\big)\, p_{z_i}^{(i)}(t)\, p_{ij}^{-} P(k+I_i+I_j,t)+$$

$$-\left(\sum_{i=1}^{n}\lambda_0 p_{0i}^{+}\left(K-\sum_{i=1}^{n}k_i(t)\right)+\sum_{i=1}^{n}\lambda_0 p_{0i}^{-}\left(K-\sum_{i=1}^{n}k_i(t)\right)+\right.$$

$$+\sum_{i,j=1}^{n}\sum_{z_i=0}^{m_i}\mu_i \min\big(k_i(t), z_i(t)\big)\, p_{z_i}^{(i)}(t)\, p_{ij}^{-}(1-\theta(k_j(t)))+$$

$$\left.+\sum_{i=1}^{n}\sum_{z_i=0}^{m_i}\mu_i \min\big(k_i(t), z_i(t)\big)\, p_{z_i}^{(i)}(t)\right)P(k,t).$$

The last equation cannot be solved analytically for a large n. In connection with this, irrespective of the time t, consider the important asymptotic case of a large number of customers $K \gg 1$ [13, 14]. Suppose that we are interested in the properties of the process $k(t)$ as K becomes very large. The vector of relative variables $\xi(t)=k(t)K^{-1}$ in a short time interval undergoes a state change by e_i, where $e_i = I_i\varepsilon$, $\varepsilon = K^{-1}$. In the case when $K\to\infty$ we have $\varepsilon\to 0$ and the vector $\xi(t)$ will be the continuous-time continuous-state Markov process with a probability density function $p(x,t)$. The points x are located at the n-dimensional lattice vertex of the set $X=\left\{x=(x_1,x_2,...,x_n):\ x_i\ge 0,\ i=\overline{1,n},\ \sum_{i=1}^{n}x_i\le 1\right\}$ at a distance of ε from each other. At $K\to\infty$, the distance between the vertices decreases, $\varepsilon\to 0$, since we can also assume that the limiting distribution of $\xi(t)$ is continuous. The density $p(x,t)$ satisfies the asymptotic relation $\lim\limits_{K\to\infty}K^n P(k,t)=p(x,t),\ x\in X.$ Realizing the asymptotic transition for the previous equation, we obtain the following partial differential equation:

$$\frac{\partial p(x,t)}{\partial t}=K\sum_{i=1}^{n}\lambda_0 p_{0i}^{+}\left(1-\sum_{i=1}^{n}x_i\right)\big(p(x-e_i,t)-p(x,t)\big)+$$

$$+\sum_{i=1}^{n}\lambda_0 p_{0i}^{+} p(x-e_i,t)+K\left(\sum_{i=1}^{n}\lambda_0 p_{0i}^{-}\left(1-\sum_{i=1}^{n}x_i\right)+\right.$$

$$\left.+\sum_{i,j=1}^{n}\sum_{z_i=0}^{m_i}\mu_i\min\big(x_i,w_i\big)\,p_{z_i}^{(i)}(t)\big(p_{i0}+p_{ij}^{-}(1-\theta(x_j))\big)\right)\big(p(x+e_i,t)-p(x,t)\big)+$$

$$+\left(\sum_{i=1}^{n}\lambda_0 p_{0i}^{-}+\sum_{i,j=1}^{n}\sum_{z_i=0}^{m_i}\mu_i\frac{\partial\min\big(x_i,w_i\big)}{\partial x_i}\,p_{z_i}^{(i)}(t)\big(p_{i0}+p_{ij}^{-}(1-\theta(x_j))\big)\right)p(x+e_i,t)+$$

$$+K\sum_{i,j=1}^{n}\sum_{z_i=0}^{m_i}\mu_i \min\left(x_i, w_i\right) p_{z_i}^{(i)}(t) p_{ij}^{+}\left(p(x+e_i-e_j,t)-p(x,t)\right)+$$

$$+\sum_{i,j=1}^{n}\sum_{z_i=0}^{m_i}\mu_i \frac{\partial \min\left(x_i, w_i\right)}{\partial x_i} p_{z_i}^{(i)}(t) p_{ij}^{+} p(x+e_i-e_j,t)+$$

$$+K\sum_{i,j=1}^{n}\sum_{z_i=0}^{m_i}\mu_i \min\left(x_i, w_i\right) p_{z_i}^{(i)}(t) p_{ij}^{-}\left(p(x+e_i+e_j,t)-p(x,t)\right)+$$

$$+\sum_{i,j=1}^{n}\sum_{z_i=0}^{m_i}\mu_i \frac{\partial \min\left(x_i, w_i\right)}{\partial x_i} p_{z_i}^{(i)}(t) p_{ij}^{-} p(x+e_i+e_j,t), \quad w_i = z_i\varepsilon.$$

If $p(x,t)$ is a twice continuously differentiable function with respect to x, then we can use the Taylor series of functions $p(x \pm e_i, t)$, $p(x+e_i-e_j,t)$, $p(x+e_i+e_j,t)$, $i, j=\overline{1,n}$, formed by the first two terms. Substituting this series into the last equation and grouping terms, we obtain that $p(x,t)$ satisfies the multidimensional Fokker-Planck-Kolmogorov equation (3) with drifts (4) and diffusion coefficients (5) [13, 14, 17]. The error in this approximation is no more than ε^2. ■

5. System of differential equations for the first-order and second-order moments of the process of changing the number of customers

The characteristic function $\varphi(\lambda,t)=\int_{\mathbb{R}^n} e^{I\lambda x^{\mathrm{T}}} p(x,t)\,dx$ of a stochastic process, $I=\sqrt{-1}$, gives as much information about the process as the probability density. Multiplying both sides of (3) by $e^{I\lambda x^{\mathrm{T}}}$, then integrating over x and taking into account certain initial and boundary conditions for equation (3), we derive the equation (6) for the characteristic function [15, 16]:

$$\frac{\partial \varphi(\lambda,t)}{\partial t}=\int_{\mathbb{R}^n}\left\{\sum_{i=1}^{n} I\lambda_i A_i(x,t)-\frac{\varepsilon}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j B_{ij}(x,t)\right\} p(x,t) e^{I\lambda x^{\mathrm{T}}}\,dx. \tag{6}$$

According to one of the most important properties of the characteristic function, it was found that the system of differential equations for the first-order and second-order moments of the state vector elements $\xi_i(t)$ is

$$\frac{d\nu_i^{(1)}(t)}{dt}=\frac{dM\left(\xi_i(t)\right)}{dt}=I^{-1}\left.\frac{\partial^2\varphi(\lambda,t)}{\partial t\,\partial\lambda_i}\right|_{\lambda=0}=A_i(\nu^{(1)}(t)),\ i=\overline{1,n}; \tag{7}$$

$$\frac{dv_{ij}^{(1,1)}(t)}{dt} = \frac{dM\left(\xi_i(t)\xi_j(t)\right)}{dt} = I^{-2}\left.\frac{\partial^3\varphi(\lambda,t)}{\partial t\partial\lambda_i\,\partial\lambda_j}\right|_{\lambda=0} =$$
$$= M\left(\xi_i(t)A_j(\xi(t))\right) + M\left(\xi_j(t)A_i(\xi(t))\right) + \varepsilon B_{ij}(v^{(1)}(t)),\ i=\overline{1,n},\ j=\overline{1,n}. \qquad (8)$$

Mathematical expectation $M\left(\xi_i(t)\right)$ and variance $\sigma_{z_i}^2(t) = \sigma^2\left(\xi_i(t)\right) = = v_{ii}^{(1,1)}(t) - \left(v_i^{(1)}(t)\right)^2$ in the general case are deterministic functions of time that determine, respectively, the expected trajectory of the process and the scattering around it, $i=\overline{1,n}$. Interval $\left(v_i^{(1)}(t) \pm \sigma_i(t)\right)$ is the interval in which the implementations of process $\xi_i(t)$ fall with a probability of about 70%.

6. Numerical example

Consider the Markov G-network consisting of unreliable nodes S_i, $i=\overline{1,4}$, and source S_0. Let the total number of customers served in the network be equal to $K = 10\,000$. Let the G-network be specified by the following parameters: the non-zero elements of transition matrix are $p_{01}^{+} = 0.45$, $p_{01}^{-} = 0.05$, $p_{02}^{+} = 0.48$, $p_{02}^{-} = 0.02$, $p_{13}^{+} = 0.57$, $p_{13}^{-} = 0.1$, $p_{10} = 0.33$, $p_{24}^{+} = 0.65$, $p_{24}^{-} = 0.1$, $p_{20} = 0.25$, $p_{31}^{+} = 0.99$, $p_{31}^{-} = 0.01$, $p_{42}^{+} = 0.99$, $p_{42}^{-} = 0.01$; the arrival rate is $\lambda_0 = 0.003$; the number of servers in network nodes are $m_1 = m_2 = m_3 = m_4 = 4$; the service rates are $\mu_1 = 0.5$, $\mu_2 = 0.7$, $\mu_3 = 0.15$, $\mu_4 = 0.15$; the server failure rates in the systems are $\alpha_1 = 0.1$, $\alpha_2 = \alpha_3 = 0.2, \alpha_4 = 0.1$; the repair rates in the systems are $\beta_1 = 0.4, \beta_2 = 0.2, \beta_3 = 0.3$, $\beta_4 = 0.2$; the initial conditions are $p_j^{(i)}(0) = 0$; $v_i^{(1)}(0) = 0$, $v_{ij}^{(1,1)}(0) = 0$, $i,j=\overline{1,4}$.

First of all, the non-stationary probability distribution $p_{z_i}^{(i)}(t)$, $i=\overline{1,n}$, should be found by solving (2) with initial condition as above. The result is cumbersome, so it is not presented in the article. Obviously, there is a stationary probability distribution of the process $z_i(t)$, $i=\overline{1,n}$, which determines the average relative time during which the process is in each of the states. For example, for the node S_1 the result is $\lim\limits_{t\to\infty} p_0^{(1)}(t) = 1/341$, $\lim\limits_{t\to\infty} p_1^{(1)}(t) = 4/341$, $\lim\limits_{t\to\infty} p_2^{(1)}(t) = 16/341$, $\lim\limits_{t\to\infty} p_3^{(1)}(t) = 64/341$, $\lim\limits_{t\to\infty} p_4^{(1)}(t) = 256/341$. This means that in a steady state, all servers in system S_1 are functional $256/341$ of the total time. The calculation results show that it is most likely that in the steady state, S_3 and S_4 will have 4 functional servers, and S_2 will have from 0 to 4 functional servers with equal probabilities. The expectation and variance of the number of functional servers at the network nodes can be calculated using the probabilities $p_{z_i}^{(i)}(t)$, $i=\overline{1,n}$.

The calculation result for the node S_2 is presented in Figure 1, where the expectation $M_{z_2}(t)$ is a solid line, the limits of the interval $M_{z_2}(t) \pm \sigma_{z_2}(t)$ is the dashed line.

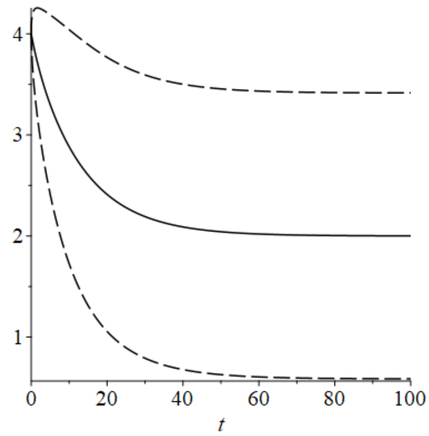

Fig. 1. The expectation and variance of the number of functional servers at S_2

Let us determine the expected number of customers in the network nodes and its variance. It is necessary to write down the system of differential equations (7), (8). Due to the form of drifts $A_i(x,t)$ and diffusion coefficients $B_{ij}(x,t)$, the solution of the system (7), (8) can be found without difficulty using numerical methods, $i, j = \overline{1,4}$. Figure 2 shows the expected trajectory of the number of customers in the node S_2 (solid line) and the variance of this number (dashed line).

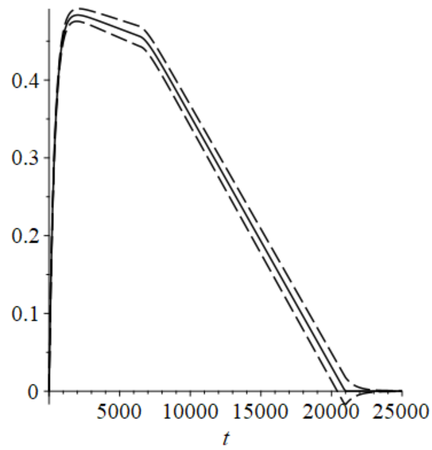

Fig. 2. The expectation and variance of the number of customers at S_2

Numerical and graphical results for other nodes can be obtained in a similar way. Analyzing the results, we see that each of the nodes requires a sufficiently large transition period of time to reach the steady state in terms of mathematical expectation and variance. It was found that the longer the duration of the transition period is, the greater the number of customers in the network.

Figure 3 shows the pairwise correlation of the number of customers in the network nodes, excluding the external environment S_0. It is interesting to observe the change and the establishment of correlation in the transient state.

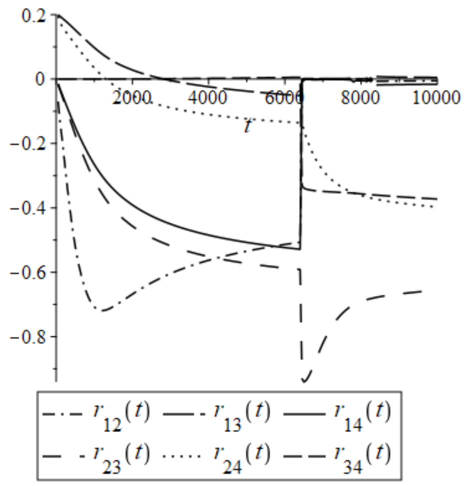

Fig. 3. The pairwise correlation of the number of customers in the network nodes

We conclude that in the steady state, the pairwise linear relationship between the number of customers can be characterized as high negative for S_2 and S_3, as moderate negative for S_2 and S_4, S_3 and S_4, and as weak for the following node pairs: S_1 and S_4, S_1 and S_3, S_1 and S_2.

7. Conclusions

In this paper, an asymptotic method was presented for studying a closed exponential G-network of unreliable queueing nodes under a limiting condition of a large number of customers. The process of changing the number of functional servers in the network nodes was modelled by the birth-death process. As a result, the expectation and the variance of the number of functional servers and of the number of customers in each network node can be found at any fixed time in both the transient and steady state. The presented technique allows us to investigate the correlation between the number of customers in different nodes with time.

The areas of implementation of research results are the design of G-networks, solving problems of their optimization and using them as models.

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