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THE HOMOTOPY ANALYSIS RANGAIG TRANSFORM METHOD FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The idea suggested in this article is to combine the Rangaig transform with the homotopy analysis method in order to facilitate the solution of nonlinear partial differential equations. This method may be called the homotopy analysis Rangaig transform method (HARTM). The proposed example results showed that HARTM is an effective method for solving nonlinear partial differential equations.

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1. Introduction

Finding solutions to nonlinear differential equations, whether ordinary or partial differential, is one of the most important difficult steps encountered by researchers in the field of mathematics or physics. As a result, we find that many researchers are striving to develop existing methods or discover new ones for this reason [1–4]. These efforts have strengthened this area of research through many methods. Among them we find the optimal homotopy asymptotic method and its applications to linear or nonlinear problems of integer order or fractional order [5–7], and the homotopy analysis method (HAM). Liao Shijun of Shanghai Jiaotong University created this method in his doctoral dissertation in 1992 [8–10].

One of the most recent contributions to this method is to combine it with some transforms such as the Laplace transform method [11], the Sumudu transform method [12], the Natural transform method [13], the Elzaki transform method [14], the Aboodh transform method [15], the Shehu transform method [16] and others. Among these are the homotopy analysis method coupled with Laplace transform [17–19], the homotopy analysis Sumudu transform method [20, 21], the homotopy analysis Natural transform method [22, 23], the homotopy analysis Elzaki transform method [24], the homotopy analysis Aboodh transform method [25] and the homotopy analysis Shehu transform method [26].

The aim of this present study is to integrate two powerful methods for solving nonlinear partial differential equations, namely the homotopy analysis method and the Rangaig transformation method [27, 28], to develop a new method. The Rangaig transform homotopy analysis method (HARTM) is the name of the updated method. Some examples are offered to demonstrate the method's usefulness.

2. Definitions and properties of the Rangaig transform

In this section, we give some basic definitions and properties of the Rangaig transform which are used further in this paper. A new transform called the Rangaig transform defined for a function of exponential order, we consider functions in the set H [27]:

$$H = \{h(t) : \exists\, N, \lambda_1, \lambda_2 > 0, |h(t)| > N e^{\lambda_i |t|}, t \in (-1)^{i-1} \times (-\infty, 0]\}, \tag{1}$$

the arbitrary constant N must be a finite and the constants λ_1, λ_2 can be infinite or infinitely finite. Introducing a new transform which is defined in (1) by:

$$R[h(t)] = T(\omega) = \frac{1}{\omega}\int_{-\infty}^{0} e^{\omega t} h(t)\mathrm{d}t,\ \frac{1}{\omega} \leq \omega \leq \frac{1}{\omega}, \tag{2}$$

is called the Rangaig transform. In this transform, the variable ω factorizes the variable t of the function h or in another sense, the function $h(t)$ is mapped into the function $T(\omega)$ of ω-space. We will summarize some results of the Rangaig transform for some functions [27, 28] in Table 1.

Table 1. Rangaig transform of some elementary functions

$h(t)$	$R[h(t)]$	$h(t)$	$R[h(t)]$
1	$\frac{1}{\omega^2}$	$\sin(t)$	$-\frac{1}{\omega(\omega^2+1)}$
t	$-\frac{1}{\omega^3}$	$\cos(t)$	$\frac{1}{\omega^2+1}$
$t^n, n \geq 0$	$\frac{(-1)^n n!}{\omega^{n+2}}$	$t^n, n \leq 0$	$\frac{(-1)^{n+1}\Gamma(-n)}{\omega^n}$
e^{at}	$\frac{1}{\omega(\omega+a)}$	$M(t-a)$	$\frac{1}{\omega^2}e^{at}$

Theorem 1 *[27](Rangaig transform of derivatives) If $h(t), h^1(t), ..., h^n(t) \in H$, then*

$$R[h^n(t)] = T(\omega) = (-1)^n \omega^n T(\omega) + (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^k \omega^{n-2-k} h^{(k)}(0). \tag{3}$$

Theorem 2 *[27](Rangaig transform of integrals) If*

$$m^n(t) = \int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\dots\int_0^{t_n+1} h(\tau)(\mathrm{d}\tau)^n,$$

such that $m(t) \in H$. Then, the Rangaig transform of $m^n(t)$ is defined as:

$$R[m^n(t)] = T_n(\omega) = (\frac{-1}{\omega})^n T(\omega).$$

Theorem 3 *[27] The Rangaig transform for the convolution identity given by:*

$$R\left[(h*g)(t)\right] = -\omega T_1(\omega) T_2(\omega),$$

where $(h*g)(t) = \int_0^t h(t-\tau)g(\tau)\mathrm{d}\tau$, $T_1(\omega)$ and $T_2(\omega)$ is the Rangaig transform of $h(t)$ and $g(t)$ respectively. □

Theorem 4 *[27](Duality). If $h(t)$ and $h(-t)$ exist over H, then the relation of Rangaig transform $T(\omega)$ and Laplace transform $F(\omega)$ of $h(t)$ is*

$$T(\omega) = \frac{1}{\omega}F(-\omega),$$

Proposition 1 *If $\frac{\partial u(x,t)}{\partial t}$ exist, and by using integration by parts, we obtain:*

$$R\left[\frac{\partial u(x,t)}{\partial t}\right] = -\omega T(x,\omega) + \frac{1}{\omega}u(x,0). \tag{4}$$

PROOF We utilize the integration by parts and the formula (2) to demonstrate it. ■

Proposition 2 *Let $T(x,\omega)$ be the Rangaig transform of $u(x,t)$, then one has:*

$$R\left[\frac{\partial^n u(x,t)}{\partial t^n}\right] = (-1)^n \omega^n T(x,\omega) + (-1)^{n+1}\sum_{k=0}^{n-1}(-1)^k \omega^{n-2-k}\frac{\partial^k u(x,0)}{\partial t^k}. \tag{5}$$

□

PROOF To demonstrate the validity of (5), we use mathematical induction.
If $n = 1$ and according to formula (5), we obtain:

$$R\left[\frac{\partial u(x,t)}{\partial t}\right] = -\omega T(x,\omega) + \frac{1}{\omega}u(x,0). \tag{6}$$

So, according to (4), we note that the formula holds when $n = 1$.
Assume inductively that the formula holds for n, so that:

$$R\left[\frac{\partial^n u(x,t)}{\partial t^n}\right] = (-1)^n \omega^n T(x,\omega) + (-1)^{n+1}\sum_{k=0}^{n-1}(-1)^k \omega^{n-2-k}\frac{\partial^k u(x,0)}{\partial t^k}, \tag{7}$$

and show that it stays true at rank $n+1$. Let $\dfrac{\partial^n u(x,t)}{\partial t^n} = v(x,t)$ and according to (4) and (7), we have:

$$\begin{aligned}
R\left[\frac{\partial^{n+1}u(x,t)}{\partial t^{n+1}}\right] &= R\left[\frac{\partial v(x,t)}{\partial t}\right] \\
&= -\omega R[v(x,t)] + \frac{1}{\omega}v(x,0) \\
&= -\omega\left[(-1)^n\omega^n T(x,\omega) + (-1)^{n+1}\sum_{k=0}^{n-1}(-1)^k\omega^{n-2-k}\frac{\partial^k u(x,0)}{\partial t^k}\right] + \frac{1}{\omega}\frac{\partial^n u(x,t)}{\partial t^n} \\
&= (-1)^{n+1}\omega^{n+1}T(x,\omega) + (-1)^{n+2}\sum_{k=0}^{n-1}(-1)^k\omega^{n-1-k}\frac{\partial^k u(x,0)}{\partial t^k} + \frac{1}{\omega}\frac{\partial^n u(x,t)}{\partial t^n} \\
&= (-1)^{n+1}\omega^{n+1}T(x,\omega) + (-1)^{n+2}\sum_{k=0}^{n}(-1)^k\omega^{n-1-k}\frac{\partial^k u(x,0)}{\partial t^k}.
\end{aligned}$$

Thus by the principle of mathematical induction, the formula (5) holds for all $n \geq 1$.■

3. Homotopy analysis Rangaig transform method (HARTM)

We consider the following general nonlinear partial differential equation as:

$$\frac{\partial^n U(x,t)}{\partial t^n} + L[U(x,t)] + M[U(x,t)] = f(x,t), \tag{8}$$

where $n = 1,2,3,...$, with the initial conditions

$$\left[\frac{\partial^{n-1}U(x,t)}{\partial t^{n-1}}\right]_{t=0} = g_{n-1}(x),\ n = 1,2,3,..., \tag{9}$$

and $\dfrac{\partial^n U(x,t)}{\partial t^n}$ is the partial derivative of the function $U(x,t)$ of order n, L is the linear partial differential operator, M represents the general nonlinear partial differential operator, and $f(x,t)$ is the source term.

Applying the Rangaig transform on both sides of (8), we can get:

$$R\left[\frac{\partial^n U(x,t)}{\partial t^n}\right] + R\left(L[U(x,t)] + M[U(x,t)]\right) = R[f(x,t)]. \tag{10}$$

Using the property of the Rangaig transform, we have the following form:

$$R[U(x,t)] - \sum_{k=0}^{n-1}\frac{(-1)^k}{\omega^{k+2}}\frac{\partial^k U(x,0)}{\partial t^k} - \frac{1}{\omega^n}R(L[U(x,t)] + M[U(x,t)] - f(x,t)) = 0. \tag{11}$$

Define the nonlinear operator:

$$N[\phi(x,t;p)] = R[\phi(x,t;p)] - \sum_{k=0}^{n-1} \frac{(-1)^k}{\omega^{k+2}} \frac{\partial^k \phi(x,0;p)}{\partial t^k} - \frac{1}{\omega^n} R(L[\phi(x,t;p)] + M[\phi(x,t;p)] - f(x,t)). \tag{12}$$

By means of the homotopy analysis method [8–10], we construct the so-called the zero-order deformation equation:

$$(1-q)R[\phi(x,t;p) - \phi(x,t;0)] = phH(x,t)N[\phi(x,t;p)], \tag{13}$$

where p is an embedding parameter and $p \in [0,1]$, $H(x,t) \neq 0$ is an auxiliary function, $h \neq 0$ is an auxiliary parameter, R is an auxiliary linear Rangaig operator.
When $p = 0$ and $p = 1$, we have:

$$\begin{cases} \phi(x,t;0) = U_0(x,t), \\ \phi(x,t;1) = U(x,t). \end{cases} \tag{14}$$

When p increases from 0 to 1, the $\phi(x,t,p)$ varied from $U_0(x,t)$ to $U(x,t)$.
Expanding $\phi(x,t;p)$ in the Taylor series with respect to p, we have:

$$\phi(x,t;p) = U_0(x,t) + \sum_{m=1}^{+\infty} U_m(x,t)p^m, \tag{15}$$

where $U_m(x,t) = \frac{1}{m!}\left[\frac{\partial^n \phi(x,t;p)}{\partial p^n}\right]_{p=0}$.

When $p = 1$, the formula (15) becomes:

$$U(x,t) = U_0(x,t) + \sum_{m=1}^{+\infty} U_m(x,t). \tag{16}$$

Define the vectors:

$$\overrightarrow{U} = \{U_0(x,t), U_1(x,t), U_2(x,t), \ldots, U_m(x,t)\}.$$

Differentiating (13) $m-$times with respect to p, then setting $p = 0$ and finally dividing them by $m!$, we obtain the so-called m^{th} order deformation equation:

$$R[U_m(x,t) - \chi_m U_{m-1}(x,t)] = hH(x,t)\Re_m(\overrightarrow{U}_{m-1}(x,t)), \tag{17}$$

where $Re_m(U_{m-1}(x,t)) = \frac{1}{(m-1)!}\left[\frac{\partial^{m-1} N(x,t;p)}{\partial p^{m-1}}\right]_{p=0}$,

and

$$\chi_n = \begin{cases} 0 & \text{if } m \geq 1, \\ 1 & \text{if } m < 1. \end{cases}$$

Applying the inverse Rangaig transform on both sides of (17), we can obtain:

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + hR^{-1}\left[H(x,t)\Re_m(\overrightarrow{U}_{m-1}(x,t))\right]. \tag{18}$$

The m^{th} deformation equation (18) is a linear which can be easily solved. So, the solution of (8) can be written in the following form:

$$U(x,t) = \sum_{m=0}^{N} U_m(x,t). \tag{19}$$

When $N \to \infty$, we can obtain an accurate approximation solution of (8). For the proof of the convergence of the homotopy analysis method see [9].

4. Application of the HARTM

In this section, we apply the homotopy analysis method coupled with the Rangaig transform method for solving somme examples of nonlinear partial differential equations.

Example 1 First, we consider the following nonlinear KdV equation:

$$U_t + UU_x - U_{xx} = 0, \ \ and \ \ U(x,0) = x. \tag{20}$$

Applying the Rangaig transform on both sides of (20), we can get:

$$-\omega R[U(x,t)] + \frac{1}{\omega}U(x,0) + R[UU_x - U_{xx}] = 0. \tag{21}$$

From (21) and the initial condition, we have:

$$R[U(x,t)] = \frac{1}{\omega^2}x + \frac{1}{\omega}R[UU_x - U_{xx}]. \tag{22}$$

We take the nonlinear part as:

$$N[\phi(x,t;p)] = R[\phi(x,t;p)] - \frac{1}{\omega^2}x - \frac{1}{\omega}R[\phi\phi_x - \phi_{xx}]. \tag{23}$$

We construct the so-called zero-order deformation equation with assumption $H(x,t) = 1$, and we have:

$$(1-q)R[\phi(x,t;p) - \phi(x,t;0) = phN[\phi(x,t;p)]. \tag{24}$$

Therefore, we have the m^{th} order deformation equation:

$$R[U_m(x,t) - \chi_m U_{m-1}(x,t)] = h\Re_m(\overrightarrow{U}_{m-1}(x,t)), \tag{25}$$

Operating the inverse Rangaig operator on both sides of (25), we get:

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + hR^{-1}[\Re_m(\overrightarrow{U}_{m-1}(x,t))], \tag{26}$$

with,

$$\Re_m(\overrightarrow{U}_{m-1}) = R[U_{m-1}] - \frac{1}{\omega^2}(1-\chi_m)x - \frac{1}{\omega}R\left[\sum_{i=0}^{m-1} U_i(U_{m-1-i})_x + (U_{m-1})_{xx}\right], \tag{27}$$

According to (26) and (27), the formulas of the first terms are given by:

$$\begin{aligned}
U_1(x,t) &= -hR^{-1}\left(\frac{1}{\omega}R\left[U_0(U_0)_x - (U_0)_{xx}\right]\right),\\
U_2(x,t) &= (1+h)U_1(x,t) - hR^{-1}\left(\frac{1}{\omega}R\left[U_0(U_1)_x + U_1(U_0)_x - (U_1)_{xx}\right]\right),\\
U_3(x,t) &= (1+h)U_2(x,t) - hR^{-1}\left(\frac{1}{\omega}R\left[U_0(U_2)_x + U_1(U_1)_x + U_2(U_0)_x - (U_2)_{xx}\right]\right),\\
&\vdots
\end{aligned} \tag{28}$$

Using the initial condition and the formulas (28), we obtain:

$$\begin{aligned}
U_0(x,t) &= x,\\
U_1(x,t) &= (h)xt,\\
U_2(x,t) &= (h)(1+h)xt + (h^2)xt^2,\\
U_3(x,t) &= (h)(1+h)^2xt + 2(h^2)(1+h)xt^2 + (h^3)xt^3,\\
&\vdots
\end{aligned} \tag{29}$$

The other components of the (HARTM) can be determined in a similar way. Finally, the approximate solution of (20) in a series form:

$$\begin{aligned}
U(x,t) &= U_0(x,t) + U_1(x,t) + U_2(x,t) + U_3(x,t) + \cdots,\\
&= x + (h)(3+3h+h^2)xt + h^2(3+2h)xt^2 + (h^3)xt^3 + \cdots,
\end{aligned} \tag{30}$$

Substituting $h = -1$ in (30), the approximate solution of (20) is given as follows:

$$U(x,t) = x[1 - t + t^2 - t^3 + \cdots]. \tag{31}$$

And in the closed form, it is given by:

$$U(x,t) = \frac{x}{1+t}, \mid t \mid < 1. \tag{32}$$

The result represents the exact solution of the equation (20) as presented in [29].□

Example 2 Second, we consider the nonlinear gas dynamics equation:

$$U_t + UU_x - U(1-U) = 0, \quad U(x,0) = \mathrm{e}^{-x}. \tag{33}$$

By following the same previous steps and using the initial condition, the first terms of the solution are given by:

$$\begin{aligned}
&U_0(x,t) = \mathrm{e}^{-x},\\
&U_1(x,t) = (-h)\mathrm{e}^{-x}t,\\
&U_2(x,t) = (-h)(1+h)\mathrm{e}^{-x}t + (h^2)\mathrm{e}^{-x}\frac{t^2}{2!},\\
&U_3(x,t) = (-h)(1+h)^2\mathrm{e}^{-x}t + 2(h^2)(1+h)\mathrm{e}^{-x}\frac{t^2}{2!} + (-h^3)\mathrm{e}^{-x}\frac{t^3}{3!},\\
&\quad\vdots
\end{aligned} \tag{34}$$

The other components of the (HARTM) can be determined in a similar way. Finally, the approximate solution of (33) in a series form:

$$\begin{aligned}
U(x,t) &= U_0(x,t) + U_1(x,t) + U_2(x,t) + U_3(x,t) + \cdots,\\
&= \mathrm{e}^{-x} + (-h)[2+h+(1+h)^2]\mathrm{e}^{-x}t + h^2(3+2h)\mathrm{e}^{-x}\frac{t^2}{2!} + (-h^3)\mathrm{e}^{-x}\frac{t^3}{3!} + \cdots
\end{aligned} \tag{35}$$

Substituting $h=-1$ in (35), the approximate solution of (33) is given as follows:

$$U(x,t) = \mathrm{e}^{-x}[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots]. \tag{36}$$

And in the closed form, it is given by:

$$U(x,t) = \mathrm{e}^{t-x}. \tag{37}$$

The result represents the exact solution of the equation (33) as presented in [29].□

Example 3 Finally, we consider the following nonlinear wave-like equation with variable coefficients:

$$U_{tt} = x^2\frac{\partial}{\partial x}(U_xU_{xx}) - x^2(U_{xx})^2 - U, \;\; U(x,0) = 0 \;\; \textit{and} \;\; U_t(x,0) = x^2. \tag{38}$$

By following the same previous steps, we obtain the m^{th} order deformation equation:

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + R{-}1[h\Re_m(\overrightarrow{U}_{m-1}(x,t))], \tag{39}$$

with,

$$\Re_m(\overrightarrow{U}_{m-1}) = R[U_{m-1}] - \frac{1}{\omega^3}(1-\chi_m)x^2$$
$$-\frac{1}{\omega^2}R\left[-x^2\frac{\partial}{\partial x}\sum_{i=0}^{m-1}(U_i)_x(U_{m-1-i})_{xx} + x^2\sum_{i=0}^{m-1}(U_i)_{xx}(U_{m-1-i})_{xx} + U_{m-1}\right] \quad (40)$$

According to (38) and (40), the formulas of the first terms are given by:

$$U_1 = hR^{-1}\left(\frac{1}{\omega^2}R\left[-x^2\frac{\partial}{\partial x}(U_{0x})(U_{0xx}) + x^2(U_{0xx})^2 + U_0\right]\right),$$
$$U_2 = (1+h)U_1 + hR^{-1}\left(\frac{1}{\omega^2}R\left[-x^2\frac{\partial}{\partial x}((U_{0x})(U_{1xx}) + (U_{1x})(U_{0xx})) + 2x^2(U_{0xx})(U_{1xx}) + U_1\right]\right),$$
$$U_3 = (1+h)U_2 + hR^{-1}\left(\frac{1}{\omega^2}R\left[-x^2\frac{\partial}{\partial x}((U_{0x})(U_{2xx}) + (U_{1x})(U_{1xx}) + (U_{2x})(U_{0xx}))\right]\right)$$
$$+hR^{-1}\left(\frac{1}{\omega^2}R\left[x^2 2((U_{0xx})(U_{2xx}) + (U_{1xx})^2) + U_1\right]\right),$$
$$\vdots \quad (41)$$

and so on. Consequently, while taking the initial conditions and according to (41), the first few components of the homotopy analysis Rangaig transform method of (38), are derived as follows:

$$U_0(x,t) = x^2 t,$$
$$U_1(x,t) = (h)x^2\frac{t^3}{3!},$$
$$U_2(x,t) = (h)(1+h)x^2\frac{t^3}{3!} + (h^2)x^2\frac{t^5}{5!},$$
$$U_3(x,t) = (h)(1+h)^2x^2\frac{t^3}{3!} + 2(h^2)(1+h)x^2\frac{t^5}{5!} + (h^3)x^2\frac{t^7}{7!},$$
$$\vdots \quad (42)$$

The other components of the (HARTM) can be determined in a similar way. Finally, the approximate solution of (38) in a series form is as follows:

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + U_3(x,t) + \cdots,$$
$$= x^2t + (h)(3+3h++h^2)x^2\frac{t^3}{3!} + h^2(3+2h)x^2\frac{t^5}{5!} + (h^3)x^2\frac{t^7}{7!} + \cdots \quad (43)$$

Substituting $h=-1$ in (43), we obtain:

$$U(x,t)=x^2\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\cdots\right). \tag{44}$$

And in the closed form, it is given by:

$$U(x,t)=x^2 sin(t). \tag{45}$$

The result (45) is the same as it was presented in [30]. □

5. Conclusion

The method proposed in this paper, which is the combination of the Rangaig transform method with the homotopy analysis method, has proven effective in solving nonlinear partial differential equations. The proposed algorithm provides the solution in a series form that converges rapidly to the exact solution if it exists. From the obtained results, it is clear that the HARTM yields very accurate solutions using only a few iterates. Accordingly, it can be said that the algorithm chosen through this combination is powerful and effective in its use in solving nonlinear partial differential equations and ordinary differential equation. Based on the paper [27], the author demonstrated that the Rangaig transform has a clearer and deeper connection to the Laplace transform. However, there are cases that the Laplace transform cannot solve the differential equations with the variable coefficients as $\frac{1}{t}$, but can be solved by applying the Rangaig transform (see the example 4.3 and the example 4.4 of [27]). Therefore, the Rangaig transform can be used as an effective tool in solving integro-differential equations and is also effective when combining it with other methods to solve the linear and nonlinear in integer order or fractional order with variable coefficients of the form t^n where n negative.

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