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THE VELOCITY FIELD TO UNSTEADY FLUID FLOW IN A CIRCULAR CYLINDER WITH GENERALIZED CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this paper, we study the velocity field corresponding to the unsteady flow of a second-grade fluid with a generalized Caputo fractional derivative in a circular cylinder. The analytical solution of the velocity field has been obtained utilizing the ρ-Laplace and the finite Hankel transforms. The solution is obtained in terms of a series containing the Mittag-Leffler functions, being the generalization of the exponential function. The effect of the fractional parameters α and ρ on fluid motion are illustrated graphically for three different cases. The model discussed in this work is more general and can be applied to other fluid models.

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1. Introduction

Recent advancements in fractional calculus have prominently showcased its modern applications across differential and integral equations, physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology, and electrochemistry [1-9]. Undoubtedly, fractional calculus has emerged as a dynamic and innovative mathematical tool for solving a wide array of problems in mathematics, science, and engineering [10-13]. Fractional differential equations, which involve derivatives of non-integer orders, are used to model systems with memory and long-term effects. Several methods, such as analytical techniques, numerical approaches, and approximate solutions, are available to solve these equations [14-23]. The integral transforms technique represents a systematic, efficient, and powerful tool [17-23]. Consider the dimensionless equation of an incompressible fractional second-grade fluid with a source term in a circular cylinder [12], which is given by

$$ {}_{0}^{C}D_{t}^{\alpha,\rho} v(r,t) = \gamma B(t) + \left(1 + \beta\ {}_{0}^{C}D_{t}^{\alpha,\rho}\right)\left(\frac{\partial^2 v(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r,t)}{\partial r}\right) + cv(r,t) + q(r,t), \tag{1} $$

with initial and boundary conditions

$$ v(r,0) = f(r),\ 0 \le r \le R, \tag{2} $$

$$ v(R,t) = h(t),\ t \ge 0, \tag{3} $$

where v is the velocity of flow, r is the radial coordinate, t is the time, $B(t)$ is the pressure gradient, $q(r,t)$ is the source term, γ, β, and c are constants such that γ is related to the pressure fluctuation amplitude, β is the ratio between the second-grade fluid parameter and the fluid density, whereas ${}_{0}^{C}D_{t}^{\alpha,\rho}$ is the left generalized Caputo fractional derivative defined as [18]

$$ {}_{0}^{C}D_{t}^{\alpha,\rho} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{-\alpha} f'(\tau)d\tau\,,\ 0 < \alpha < 1,\ \rho > 0, \tag{4} $$

where $\Gamma(\cdot)$ is the Gamma function.

This model is applied in simulating viscoelastic fluid flows in confined geometries, such as blood flow in arteries, polymer processing in cylindrical molds, and fluid transport in pipelines, where memory effects and nonlocal behaviors are significant [24-26].

In this paper, we use the ρ-Laplace and finite Hankel transforms to obtain the analytic solution of Eq. (1) with the conditions (2) and (3) in general form. In addition, we illustrate the effect of the fractional parameters α and ρ on the obtained solution graphically, when $B(t) = B_0 + B_1\, e^{i\omega t}$ [12] with the following three different cases:

Case 1 At $q(r,t) = 0,\ f(r) = 0,\ h(t) = 0,$ (5)

Case 2 At $q(r,t) = \dfrac{\delta(r)}{r}\delta\left(\dfrac{t^\rho}{\rho}\right),\ f(r) = 0,\ h(t) = 0,$ (6)

Case 3 At $q(r,t) = 0,\ f(r) = a(a^2 + r^2)^{\frac{-3}{2}},\ h(t) = 0,$ (7)

where B_0, B_1, and ω are constants, $\delta(r)$ and $\delta\left(\frac{t^\rho}{\rho}\right)$ are Dirac delta functions.

This paper is organized as follows: Section 2 presents foundational definitions and tools related to fractional calculus. In Section 3, the solution of Eq. (1) with the conditions (2) and (3) is investigated. In Section 4, the effect of the fractional parameters α and ρ on the velocity profile is illustrated graphically for three different cases. Finally, in Section 5, we present the conclusions of this paper.

2. Basic definitions and tools

In this section, we set some definitions and lemmas relevant to the fractional derivatives.

Definition 1 Let $f : [0,\infty) \rightarrow R$ be a real-valued function. The ρ-Laplace transform of f is defined as

$$\mathcal{L}_\rho\{f(t)\} = f^*(s) = \int_0^\infty e^{-s\frac{t^\rho}{\rho}} f(t) \frac{dt}{t^{1-\rho}}, \rho > 0, \tag{8}$$

for all values of s, where the integral is valid [18].

Theorem 1 Let $f : [0,\infty) \rightarrow R$ be a real-valued function such that its ρ-Laplace transform exists. Then

$$\mathcal{L}_\rho\left\{f\left(\frac{t^\rho}{\rho}\right)\right\} = \mathcal{L}\{f(t)\} = F(s),\ \ \mathcal{L}_\rho^{-1}\{F(s)\} = f\left(\frac{t^\rho}{\rho}\right), \tag{9}$$

where $\mathcal{L}\{f(t)\}$ is the usual Laplace transform [18].

Lemma 1 [18]

$$1)\ \mathcal{L}_\rho\left\{\delta\left(\frac{t^\rho}{\rho}\right)\right\} = 1,\ \ 2)\ \mathcal{L}_\rho\{t^\beta\} = \rho^{\frac{\beta}{\rho}} \frac{\Gamma(1+\frac{\beta}{\rho})}{s^{1+\frac{\beta}{\rho}}},\ \ \beta \in R,\ \ s > 0.$$

Lemma 2 Let $f : [0,\ \infty) \rightarrow R$ be a real-valued function such that its ρ-Laplace transform exists. Then [18]

$$\mathcal{L}_\rho\{{}_0^C D_t^{\alpha,\rho} f(t)\} = s^\alpha \mathcal{L}_\rho\{f(t)\} - s^{\alpha-1} f(0). \tag{10}$$

Definition 2 Let f and g be two functions that are piecewise continuous at each interval $[0,T]$ and of exponential order. We define the ρ-convolution of f and g by [18]

$$\big(f *_\rho g\big)(t) = \int_0^t f\left((t^\rho - \tau^\rho)^{\frac{1}{\rho}}\right) g(\tau) \frac{d\tau}{\tau^{1-\rho}}. \tag{11}$$

Theorem 2 Let f and g be two functions that are piecewise continuous at each interval $[0,\ T]$ and of exponential order $e^{c\frac{t\rho}{\rho}}$. Then [18]

$$\mathcal{L}_\rho\{f *_\rho g\} = \mathcal{L}_\rho\{f\}\mathcal{L}_\rho\{g\},\ \ s > c. \tag{12}$$

Lemma 3 Let $Re(\alpha) > 0$ and $\left|\frac{\lambda}{s^\alpha}\right| < 1$. Then [18]

$$\mathcal{L}_\rho^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha-\lambda}\right\} = \left(\frac{t^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\beta}\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right), \tag{13}$$

where $E_{\alpha,\beta}(z)$ is the two parameter Mittag-Leffler function that is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)},\ z \in C,\ Re(\alpha) > 0. \tag{14}$$

Proof

$$\begin{aligned}
\mathcal{L}_\rho^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha-\lambda}\right\} &= \mathcal{L}_\rho^{-1}\left\{\frac{1}{s^\beta}\sum_{k=0}^{\infty}\left(\frac{\lambda}{s^\alpha}\right)^k\right\} \\
&= \mathcal{L}_\rho^{-1}\left\{\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+\beta)\rho^{k\alpha+\beta-1}}\rho^{k\alpha+\beta-1}\frac{\Gamma(k\alpha+\beta)}{s^{k\alpha+\beta}}\right\} \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+\beta)\rho^{k\alpha+\beta-1}}\mathcal{L}_\rho^{-1}\left\{\rho^{k\alpha+\beta-1}\frac{\Gamma(k\alpha+\beta)}{s^{k\alpha+\beta}}\right\} \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+\beta)\rho^{k\alpha+\beta-1}}t^{\rho(k\alpha+\beta-1)} \\
&= \left(\frac{t^\rho}{\rho}\right)^{\beta-1}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+\beta)}\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)^k \\
&= \left(\frac{t^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\beta}\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right).
\end{aligned}$$

Definition 3 The finite Hankel transform of order n of a function $f(r)$ in the interval $r \in [0, R]$ is defined as

$$H_n\{f(r)\} = \bar{f}_n(k_i) = \int_0^R rf(r)J_n(rk_i)dr, \tag{15}$$

where $k_i(0 < k_1 < k_2 < \cdots)$ are the roots of the equation $J_n(Rk_i) = 0$, and J_n is the Bessel function of the first kind and n-order [21].

Definition 4 The inverse finite Hankel transform is defined by [27]

$$H_n^{-1}\{\bar{f}_n(k_i)\} = f(r) = \frac{2}{R^2}\sum_{i=1}^{\infty}\bar{f}_n(k_i)\frac{J_n(rk_i)}{J_{n+1}^2(Rk_i)}. \tag{16}$$

Lemma 4 [27]

i. $$H_n\{f'(r)\} = \frac{k_i}{2n}[(n-1)H_{n+1}\{f(r)\} - (n+1)H_{n-1}\{f(r)\}],\ \ n \geq 1, \tag{17}$$

provided $f(r)$ is finite at $r = 0$.

ii. When $n = 0$,

$$H_0\left\{f''(r) + \frac{1}{r}f'(r)\right\} = -k_i^2 \bar{f}_0(k_i) + Rk_i f(R) J_1(Rk_i). \tag{18}$$

Lemma 5 [27] The following identities hold true:

$$1)\ H_0\left\{\frac{\delta(r)}{r}\right\} = 1. \quad 2)\ H_0\left\{a(a^2 + r^2)^{\frac{-3}{2}}\right\} = e^{-ak_i}.$$

3. Solution procedure

In this section, we will determine the solution for the fractional differential Eq. (1) along with corresponding initial and boundary conditions, Eqs. (2) and (3). To do this, we use the finite Hankel transform of order zero with respect to the radial coordinate r and the ρ-Laplace transform with respect to the time variable t.

Applying the zeroth-order finite Hankel transform to Eq. (1), and using Eq. (18), we have

$$\begin{aligned} {}_0^C D_t^{\alpha,\rho} \bar{v}_0(k_i,t) = {} & \frac{\gamma}{k_i} RB(t) J_1(Rk_i) \\ & + \left(1 + \beta\ {}_0^C D_t^{\alpha,\rho}\right)\left(-k_i^2 \bar{v}_0(k_i,t) + Rk_i v(R,t) J_1(Rk_i)\right) \\ & + c\bar{v}_0(k_i,t) + \bar{q}_0(k_i,t). \end{aligned} \tag{19}$$

Using the boundary condition Eq. (3), Eq. (19) becomes

$$\begin{aligned} {}_0^C D_t^{\alpha,\rho} \bar{v}_0(k_i,t) = {} & \frac{\gamma}{k_i} RB(t) J_1(Rk_i) \\ & + \left(1 + \beta\ {}_0^C D_t^{\alpha,\rho}\right)\left(-k_i^2 \bar{v}_0(k_i,t) + Rk_i h(t) J_1(Rk_i)\right) \\ & + c\bar{v}_0(k_i,t) + \bar{q}_0(k_i,t). \end{aligned} \tag{20}$$

Applying the ρ-Laplace transform to Eq. (20), and using Eq. (10), we get

$$\begin{aligned} & s^\alpha \bar{v}_0^*(k_i,s) - s^{\alpha-1}\bar{v}_0(k_i,0) \\ & \quad = \frac{\gamma}{k_i} RJ_1(Rk_i) B^*(s) - k_i^2 \bar{v}_0^*(k_i,s) \\ & \quad - \beta k_i^2 \left(s^\alpha \bar{v}_0^*(k_i,s) - s^{\alpha-1}\bar{v}_0(k_i,0)\right) + Rk_i J_1(Rk_i) h^*(s) \\ & \quad + \beta Rk_i J_1(Rk_i)\left(s^\alpha h^*(s) - s^{\alpha-1}h(0)\right) + c\bar{v}_0^*(k_i,s) \\ & \quad + \bar{q}_0^*(k_i,s). \end{aligned} \tag{21}$$

Applying the zeroth-order finite Hankel transform to the condition Eq. (2), we have

$$\bar{v}_0(k_i,0)=\bar{f}_0(k_i). \tag{22}$$

Substituting Eq. (22) into Eq. (21), we get

$$\begin{aligned} s^{\alpha}\bar{v}_0^*(k_i,s)-s^{\alpha-1}\bar{f}_0(k_i) &= \frac{\gamma}{k_i}RJ_1(Rk_i)B^*(s)-k_i^2\bar{v}_0^*(k_i,s) \\ &-\beta k_i^2\left(s^{\alpha}\bar{v}_0^*(k_i,s)-s^{\alpha-1}\bar{f}_0(k_i)\right)+Rk_iJ_1(Rk_i)h^*(s) \\ &+\beta Rk_iJ_1(Rk_i)\left(s^{\alpha}h^*(s)-s^{\alpha-1}h(0)\right)+c\bar{v}_0^*(k_i,s) \\ &+\bar{q}_0^*(k_i,s). \end{aligned} \tag{23}$$

Rearranging and writing Eq. (23) in a more suitable form, we obtain

$$\begin{aligned} \bar{v}_0^*(k_i,s) &= \frac{\gamma RJ_1(Rk_i)}{k_i\left(1+\beta k_i^2\right)}\left[\frac{B^*(s)}{s^{\alpha}-\left(\frac{c-k_i^2}{1+\beta k_i^2}\right)}\right]+\bar{f}_0(k_i)\left[\frac{s^{\alpha-1}}{s^{\alpha}-\left(\frac{c-k_i^2}{1+\beta k_i^2}\right)}\right] \\ &+\frac{R\beta k_iJ_1(Rk_i)}{\left(1+\beta k_i^2\right)}h^*(s) \\ &+\frac{Rk_iJ_1(Rk_i)(1+\beta c)}{\left(1+\beta k_i^2\right)^2}\left[\frac{h^*(s)}{s^{\alpha}-\left(\frac{c-k_i^2}{1+\beta k_i^2}\right)}\right] \\ &-\frac{R\beta k_iJ_1(Rk_i)}{\left(1+\beta k_i^2\right)}h(0)\left[\frac{s^{\alpha-1}}{s^{\alpha}-\left(\frac{c-k_i^2}{1+\beta k_i^2}\right)}\right] \\ &+\frac{1}{\left(1+\beta k_i^2\right)}\left[\frac{\bar{q}_0^*(k_i,s)}{s^{\alpha}-\left(\frac{c-k_i^2}{1+\beta k_i^2}\right)}\right]. \end{aligned} \tag{24}$$

The inverse ρ-Laplace transform is applied to Eq. (24), which yields

$$\begin{aligned}
&\bar{v}_0(k_i,t) \\
&= \frac{\gamma R J_1(Rk_i)}{k_i(1+\beta k_i^2)}\left[B(t) *_\rho \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{c-k_i^2}{1+\beta k_i^2}\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right] \\
&+ \bar{f}_0(k_i)E_\alpha\left(\frac{c-k_i^2}{1+\beta k_i^2}\left(\frac{t^\rho}{\rho}\right)^\alpha\right) + \frac{R\beta k_i J_1(Rk_i)}{(1+\beta k_i^2)}h(t) \\
&+ \frac{Rk_i J_1(Rk_i)(1+\beta c)}{(1+\beta k_i^2)^2}\left[h(t) *_\rho \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{c-k_i^2}{1+\beta k_i^2}\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right] \\
&- \frac{R\beta k_i J_1(Rk_i)}{(1+\beta k_i^2)}h(0)E_\alpha\left(\frac{c-k_i^2}{1+\beta k_i^2}\left(\frac{t^\rho}{\rho}\right)^\alpha\right) \\
&+ \frac{1}{(1+\beta k_i^2)}\left[\bar{q}_0(k_i,t) *_\rho \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{c-k_i^2}{1+\beta k_i^2}\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right].
\end{aligned} \tag{25}$$

Finally, applying the inverse Hankel transform to Eq. (25), the solution of Eq. (1) can be expressed as

$$\begin{aligned}
&v(r,t) \\
&= \frac{2\gamma}{R}\sum_{i=1}^{\infty}\frac{J_0(rk_i)}{k_i(1+\beta k_i^2)J_1(Rk_i)}\left[B(t) *_\rho \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{c-k_i^2}{1+\beta k_i^2}\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right] \\
&+ \frac{2}{R^2}\sum_{i=1}^{\infty}\frac{J_0(rk_i)}{J_1^2(Rk_i)}\bar{f}_0(k_i)E_\alpha\left(\frac{c-k_i^2}{1+\beta k_i^2}\left(\frac{t^\rho}{\rho}\right)^\alpha\right) + \frac{2\beta}{R}\sum_{i=1}^{\infty}\frac{k_i J_0(rk_i)h(t)}{(1+\beta k_i^2)J_1(Rk_i)} \\
&+ \frac{2}{R}\sum_{i=1}^{\infty}\frac{k_i(1+\beta c)J_0(rk_i)}{(1+\beta k_i^2)^2 J_1(Rk_i)}\left[h(t) *_\rho \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{c-k_i^2}{1+\beta k_i^2}\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right] \\
&- \frac{2\beta}{R}\sum_{i=1}^{\infty}\frac{k_i h(0)J_0(rk_i)}{(1+\beta k_i^2)J_1(Rk_i)}E_\alpha\left(\frac{c-k_i^2}{1+\beta k_i^2}\left(\frac{t^\rho}{\rho}\right)^\alpha\right) \\
&+ \frac{2}{R^2}\sum_{i=1}^{\infty}\frac{J_0(rk_i)}{(1+\beta k_i^2)J_1^2(Rk_i)}\left[\bar{q}_0(k_i,t) *_\rho \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{c-k_i^2}{1+\beta k_i^2}\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right].
\end{aligned} \tag{26}$$

4. Graphical illustration

In this section, we present a graphic illustration of the effect of the fractional parameters α and ρ on the velocity profile Eq. (26) when $\gamma = 0.4, \beta = 0.55, c = 0$, $R = 1, \omega = \frac{\pi}{7}, B_0 = 0.7, B_1 = 0.8$, and $a = 5$ for the three different cases mentioned above.

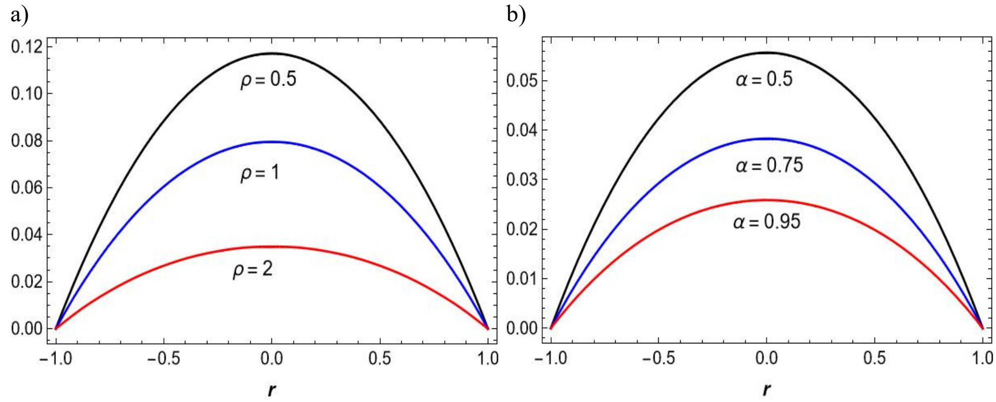

Fig. 1. The velocity flow (26) for case 1 when: a) $t = 0.5$ and $\alpha = 0.8$ at different values of ρ, b) $t = 0.5$ and $\rho = 2$ at different values of α

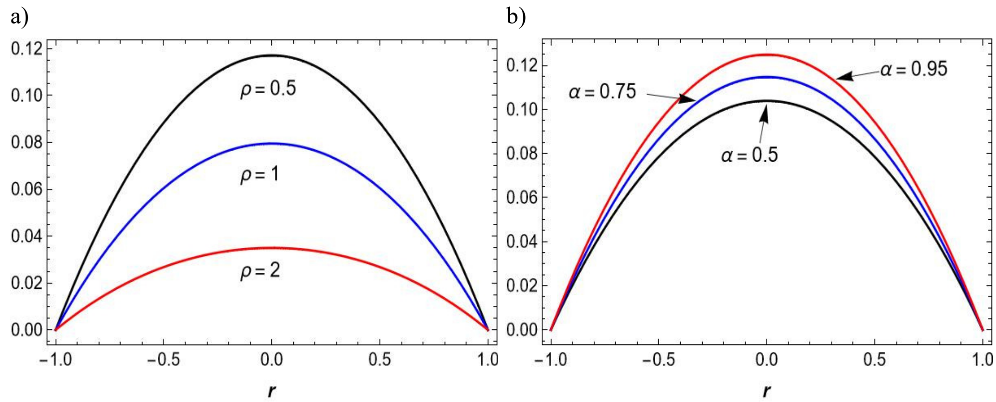

Fig. 2. The velocity flow (26) for case 2 when: a) $t = 0.5$ and $\alpha = 0.8$ at different values of ρ, b) $t = 0.5$ and $\rho = 0.5$ at different values of α

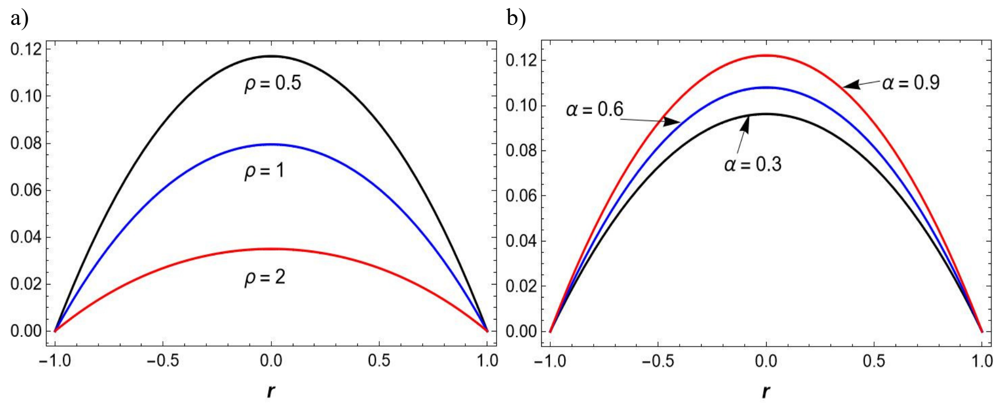

Fig. 3. The velocity flow (26) for case 3 when: a) $t = 0.5$ and $\alpha = 0.8$ at different values of ρ, b) $t = 0.5$ and $\rho = 0.5$ at different values of α

Figures 1-3 show the absolute value of the velocity flow at different values of α and ρ for the three cases mentioned above, respectively. In Figures 1a, 2a and 3a, we can notice that with increasing the value of ρ the flow velocity decreases. In Figure 1b, as the value of α increases, the flow velocity decreases, while in Figures 2b and 3b, as the value of α increases, the flow velocity increases.

5. Conclusions

The ρ-Laplace and the finite Hankel transforms are regarded as powerful techniques for solving fractional differential equations. The dimensionless equation of an incompressible fractional second-grade fluid in a circular cylinder has been investigated to obtain the exact solution for the velocity field. The fractional derivative is taken as the generalized Caputo, which is beneficial in many studies and is widely accepted. The obtained solution is expressible in terms of a series involving a new bivariate Mittag-Leffler function defined very recently and is already being discovered in various applications. The solution obtained in [12] can be considered as a special solution of our result, which is case 1 in our model. The solution is illustrated graphically for three different cases to demonstrate the influence of the fractional parameters α and ρ on the velocity profile.

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