NUMERICAL SOLUTION OF THE INVERSE PARAMETRIC PROBLEM USING THE BOUNDARY ELEMENT METHOD

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Abstract. In the paper the application of the boundary element method for numerical solution of the inverse parametric problem is presented, this means the temperature dependent thermal conductivity is identified. The thermal conductivity of the material appears in the fundamental Fourier law and also in the basic energy equation (the Kirchhoff-Fourier equation). This parameter is determined on the basis of physical experiments. The numerical identification of the thermal conductivity is possible on the basis of the knowledge of temperature field in the domain considered.

1. Formulation of the direct problem

The following 2D problem is considered

$$x \in \Omega: \quad \nabla [\lambda (T) \nabla T(x)] = 0 \tag{1}$$

where T is the temperature, $x = (x_1, x_2)$ are the spatial coordinates, λ is the thermal conductivity, $T_b(x)$ is the known boundary temperature. We assume that

$$\lambda(T) = aT^2 + bT + c \tag{2}$$

where the coefficients a, b, c in the direct problem are known.
We introduce the Kirchhoff transformation

$$U(T) = \int_0^T \lambda(\mu)\, \mathrm{d}\mu \tag{3}$$

and then the governing equations (1) take a form

$$\begin{cases} x \in \Omega: & \nabla^2 U(x) = 0 \\ x \in \Gamma: & U(x) = U_b(x) = U(T_b) \end{cases} \tag{4}$$

where

$$U(x) = \frac{1}{3} aT^3(x) + \frac{1}{2} bT^2(x) + cT(x) \tag{5}$$

2. Method of the direct problem solution

In order to solve the problem describing by equations (4) the boundary element method is used. The integral equation corresponding to the problem (4) is the following [1, 2]

$$\xi \in \Gamma: \quad \mathrm{B}(\xi)\, U(\xi) + \int_{\Gamma} \mathrm{Q}(x)\, \mathrm{U}^*(\xi, \mathrm{x})\, \mathrm{d}\Gamma = \int_{\Gamma} \mathrm{U}(x)\, \mathrm{Q}^*(\xi, \mathrm{x})\, \mathrm{d}\Gamma \tag{6}$$

where ξ is the observation point, $U^*(\xi, x)$ is the fundamental solution and for 2D domain oriented in Cartesian coordinate system it is a function of the form

$$U^*(\xi, x) = \frac{1}{2\pi} \ln \frac{1}{r} \tag{7}$$

while r is the distance between the points ξ and x. In equation (6):

$$Q(x) = -\frac{\partial U(x)}{\partial n} = -\lambda(T) \frac{\partial T(x)}{\partial n} = \lambda(T)\, W(x) \tag{8}$$

and

$$Q^*(\xi, x) = -\frac{\partial U^*(\xi, x)}{\partial n} \tag{9}$$

while $B(\xi)$ is the coefficient from the scope (0, 1).
The function $Q^*(\xi, x)$ is calculated in analytic way:

$$Q^*(\xi, x) = \frac{(x_1 - \xi_1)\cos\alpha_1 + (x_2 - \xi_2)\cos\alpha_2}{2\pi r^2} \tag{10}$$

where $\cos\alpha_1$, $\cos\alpha_2$ are the directional cosines of the normal outward vector n.
Taking into account the dependence (8), the boundary integral equation (6) can be expressed as follows

$$\mathrm{B}(\xi)\, U(\xi) + \int_{\Gamma} \lambda(T)\, W(x)\, U^*(\xi, x)\, \mathrm{d}\Gamma = \int_{\Gamma} U(x)\, Q^*(\xi, x)\, \mathrm{d}\Gamma \tag{11}$$

In numerical realization the boundary Γ is discretized. If N constant boundary elements Γ_j, $j = 1,...,N$ is used then the approximation of equation (6) takes a form

$$\frac{1}{2} U(\xi^i) + \sum_{j=1}^{N} \lambda_j\, W_j \int_{\Gamma_j} U^*(\xi^i, x)\, \mathrm{d}\Gamma_j = \sum_{j=1}^{N} U_j \int_{\Gamma_j} Q^*(\xi^i, x)\, \mathrm{d}\Gamma_j \tag{12}$$

or

$$\sum_{j=1}^{N} G_{ij}\,\lambda_j W_j = \sum_{j=1}^{N} H_{ij}\ U_j \tag{13}$$

where

$$G_{ij} = \int_{\Gamma_j} U^*(\xi^i, x)\,\mathrm{d}\Gamma_j \tag{14}$$

while

$$H_{ij} = \begin{cases} \int_{\Gamma_j} Q^*(\xi^i, x), & i \neq j \\ -0.5, & i = j \end{cases} \tag{15}$$

and $U_j = U(x^j)$, $Q_j = Q(x^j)$.

After the solution of the system of equations (13) for $i = 1,2,...,N$, the values of function U at the internal points are calculated using the formula

$$U_i = \sum_{j=1}^{N} H_{ij}\ U_j - \sum_{j=1}^{N} G_{ij}\,\lambda_j\,W_j \tag{16}$$

where $i = N+1, N+2,...,N+L$.
The obtained internal values of function U should be re-counted

$$\frac{1}{3}aT_i^3 + \frac{1}{2}bT_i^2 + cT_i - U_i(T) = 0 \tag{17}$$

One of the roots of this equation corresponds to the searched value of temperature T_i.

3. Formulation of the inverse problem and the method of solution

In the parametric inverse problem we assume that the coefficients a, b, c appearing in equation (2) are unknown, but the value of thermal conductivity for temperature T_d is known, namely $\lambda_d = \lambda(T_d)$ and also two temperatures at internal points, this means $T_d^1 = T(x^1)$ and $T_{d\,2} = T(x^2)$ are given. The aim of investigations is to determine the values of a, b, c in the equation (2) [3].

Taking into account the dependence (5), the system of equations (13) can be written in the form

$$\sum_{j=1}^{N} G_{ij}\left(aT_j^2 + bT_j + c\right)W_j = \sum_{j=1}^{N} H_{ij}\left(\frac{1}{3}aT_j^3 + \frac{1}{2}bT_j^2 + cT_j\right) \tag{18}$$

One can notice that in this system N+3 unknowns appear, this means the boundary values of W_j for $j = 1,2,...,N$ and three coefficients a, b, c. In order to solve the problem, additional three equations are needed. Two equations are connected with the knowledge of temperature at the two internal points $T_{d\,i}$ and then (c.f. equations (15) and (5))

$$\frac{1}{3}aT_{d\,\mathrm{i}}^3 + \frac{1}{2}bT_{d\,\mathrm{i}}^2 + cT_{d\,\mathrm{i}} = \sum_{j=1}^{N} H_{ij}\left(\frac{1}{3}aT_j^3 + \frac{1}{2}bT_j^2 + cT_j\right) - \sum_{j=1}^{N} G_{ij}\left(aT_j^2 + bT_j + c\right)W_j, \quad i = N+1, N+2 \tag{19}$$

The last equation results from the assumption that the value of thermal conductivity for one value of temperature is known, namely

$$\lambda_d = \lambda(T_d) = a\,T_d^2 + bT_d + c \tag{20}$$

In this way one obtains the nonlinear system of equations (18), (19), (20) which can be written in the form

$$f_i(W_1, W_2, ..., W_N, a, b, c) = 0, \quad i = 1, 2, ..., N+3 \tag{21}$$

where
- for $i = 1,2,...,N$:

$$f_i = a\sum_{j=1}^{N} G_{ij}\,T_j^{\,2}\,W_j + b\sum_{j=1}^{N} G_{ij}\,T_j\,W_j + c\sum_{j=1}^{N} G_{ij}\,W_j - \frac{1}{3}a\sum_{j=1}^{N} H_{ij}\,T_j^{\,3} - \frac{1}{2}b\sum_{j=1}^{N} H_{ij}\,T_j^{\,2} - c\sum_{j=1}^{N} H_{ij}\,T_j \tag{22}$$

- for $i = N$+1, N+2:

$$f_i = a\sum_{j=1}^{N} G_{ij}T_j^2W_j + b\sum_{j=1}^{N} G_{ij}T_j\,W_j + c\sum_{j=1}^{N} G_{ij}W_j + \frac{1}{3}a\left(T_{d\,i}^3 - \sum_{j=1}^{N} H_{ij}T_j^3\right) + \frac{1}{2}b\left(T_{d\,i}^2 - \sum_{j=1}^{N} H_{ij}T_j^2\right) + c\left(T_{d\,i} - \sum_{j=1}^{N} H_{ij}T_j\right) \tag{23}$$

- for $i = N$+3:

$$f_i = aT_d^2 + bT_d + c - \lambda_d \tag{24}$$

or

$$f_i = \begin{cases} a\sum_{j=1}^{N} G_{ij}T_j^2 W_j + b\sum_{j=1}^{N} G_{ij}T_j W_j + c\sum_{j=1}^{N} G_{ij}W_j + \\ A_i a + B_i b + C_i c, \qquad \mathrm{i} = 1,2,...,\mathrm{N} \\ a\sum_{j=1}^{N} G_{ij}T_j^2 W_j + b\sum_{j=1}^{N} G_{ij}T_j W_j + c\sum_{j=1}^{N} G_{ij}W_j + \\ \left(\frac{1}{3}T_{d\,\mathrm{i}}^3 + A_i\right)a + \left(\frac{1}{2}T_{d\,\mathrm{i}}^2 + B_i\right)b + \left(T_{d\,\mathrm{i}} + C_i\right)c, \quad i = N+1, N+2 \\ aT_d^2 + bT_d + c - \lambda_d, \quad i = N+3 \end{cases} \tag{25}$$

where:

$$A_i = -\frac{1}{3}\sum_{j=1}^{N} H_{ij}\ T_j^3 \tag{26}$$

$$B_i = -\frac{1}{2}\sum_{j=1}^{N} H_{ij}\ T_j^2 \tag{27}$$

$$C_i = -\sum_{j=1}^{N} H_{ij}\ T_j \tag{28}$$

In order to solve the system of equations (21) the Newton method is used [4]. At first, the functions f_i are differentiated with respect to the unknowns. So, for $j =$ $= 1,2,...,N$ one has

$$\frac{\partial f_i}{\partial W_j} = G_{ij}\,(aT_j^2 + bT_j + c), \; i = 1,2,...,N+2 \tag{29}$$

$$\frac{\partial f_i}{\partial b} = \begin{cases} \sum_{j=1}^{N} G_{ij}T_j W_j + B_i, & i = 1,2,...,N \\ \sum_{j=1}^{N} G_{ij}T_j W_j + \frac{1}{2}T_{d\,\mathrm{i}}^2 + B_i, & i = N+1, N+2 \end{cases} \tag{30}$$

$$\frac{\partial f_i}{\partial b} = \begin{cases} \sum_{j=1}^{N} G_{ij}T_j W_j + B_i, & i = 1,2,...,N \\ \sum_{j=1}^{N} G_{ij}T_j W_j + \frac{1}{2}T_{d\,\mathrm{i}}^2 + B_i, & i = N+1, N+2 \end{cases} \tag{31}$$

$$\frac{\partial f_i}{\partial c} = \begin{cases} \sum_{j=1}^{N} G_{ij} W_j + C_i, & i = 1,2,..., N \\ \sum_{j=1}^{N} G_{ij} W_j + T_{d\,i} + C_i, & i = N+1, N+2 \end{cases} \tag{32}$$

$$\begin{aligned} &\frac{\partial f_{N+3}}{\partial W_j} = 0, \quad j = 1,2,...,N \\ &\frac{\partial f_{N+3}}{\partial a} = T_d^2, \quad \frac{\partial f_{N+3}}{\partial b} = T_d, \quad \frac{\partial f_{N+3}}{\partial c} = 1 \end{aligned} \tag{33}$$

For the arbitrary assumed values of $W_i = W_i^m$, $i = 1,..., N$, $a = a^m$, $b = b^m$, $c = c^m$, the values of functions f_i, the values of derivatives $\partial f_i / \partial W_j$ for $i = 1,2,...,N+3$, $j = 1,2,..., N$ and the values of $\partial f_i / \partial a$, $\partial f_i / \partial b$, $\partial f_i / \partial c$ for $i = 1,2,...,N+3$ are calculated. Next, the linear system of equations must be solved

$$\begin{bmatrix} \frac{\partial f_1}{\partial W_1} & \frac{\partial f_1}{\partial W_2} & \cdots & \frac{\partial f_1}{\partial W_N} & \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} & \frac{\partial f_1}{\partial c} \\ \frac{\partial f_2}{\partial W_1} & \frac{\partial f_2}{\partial W_2} & \cdots & \frac{\partial f_2}{\partial W_N} & \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} & \frac{\partial f_2}{\partial c} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_N}{\partial W_1} & \frac{\partial f_N}{\partial W_2} & \cdots & \frac{\partial f_N}{\partial W_N} & \frac{\partial f_N}{\partial a} & \frac{\partial f_N}{\partial b} & \frac{\partial f_N}{\partial c} \\ \frac{\partial f_{N+1}}{\partial W_1} & \frac{\partial f_{N+1}}{\partial W_2} & \cdots & \frac{\partial f_{N+1}}{\partial W_N} & \frac{\partial f_{N+1}}{\partial a} & \frac{\partial f_{N+1}}{\partial b} & \frac{\partial f_{N+1}}{\partial c} \\ \frac{\partial f_{N+2}}{\partial W_1} & \frac{\partial f_{N+2}}{\partial W_2} & \cdots & \frac{\partial f_{N+2}}{\partial W_N} & \frac{\partial f_{N+2}}{\partial a} & \frac{\partial f_{N+2}}{\partial b} & \frac{\partial f_{N+2}}{\partial c} \\ \frac{\partial f_{N+3}}{\partial W_1} & \frac{\partial f_{N+3}}{\partial W_2} & \cdots & \frac{\partial f_{N+3}}{\partial W_N} & \frac{\partial f_{N+3}}{\partial a} & \frac{\partial f_{N+3}}{\partial b} & \frac{\partial f_{N+3}}{\partial c} \end{bmatrix} \begin{bmatrix} \Delta W_1^m \\ \Delta W_2^m \\ \cdots \\ \Delta W_N^m \\ \Delta a^m \\ \Delta b^m \\ \Delta c^m \end{bmatrix} =$$

$$= - \begin{bmatrix} f_1(W_1^m, W_2^m, ..., W_N^m, a^m, b^m, c^m) \\ f_2(W_1^m, W_2^m, ..., W_N^m, a^m, b^m, c^m) \\ .... \\ f_N(W_1^m, W_2^m, ..., W_N^m, a^m, b^m, c^m) \\ f_{N+1}(W_1^m, W_2^m, ..., W_N^m, a^m, b^m, c^m) \\ f_{N+2}(W_1^m, W_2^m, ..., W_N^m, a^m, b^m, c^m) \\ f_{N+3}(W_1^m, W_2^m, ..., W_N^m, a^m, b^m, c^m) \end{bmatrix} \tag{34}$$

and

$$
\begin{aligned}
W_i^{m+1} &= W_i^m + \Delta W_i^m \,, \quad i = 1,2,\dots,N \\
a^{m+1} &= a^m + \Delta a^m \\
b^{m+1} &= b^m + \Delta b^m \\
c^{m+1} &= c^m + \Delta c^m
\end{aligned}
\tag{35}
$$

The iteration procedure is repeated until the accuracy required is achieved.

4. Example of computations

The square domain of dimensions 0.1×0.1 [m] has been considered. On the right and upper surfaces the boundary temperature 600°C is assumed, on the remaining parts of the boundary the temperature 100°C has been accepted. The boundary has been divided into 40 constant boundary elements.

At first the direct problem has been solved. The values of thermal conductivity of copper for different temperatures are collected in the Table 1 [5].

Table 1. Thermal conductivity of copper [5]

T, °C	0	100	200	400	600
λ, W/mK	386.116	379.138	373.323	364.019	353.552

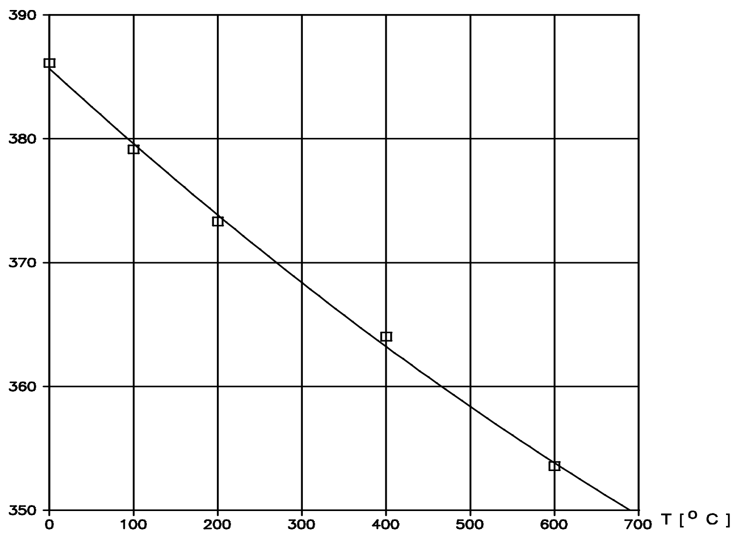

Fig 1. Thermal conductivity

Using the method of function approximation [6] the following dependence has been obtained (Fig. 1)

$$\lambda(T) = 1.525 \cdot 10^{-5} T^2 - 0.06622T + 385.661 \tag{36}$$

Next, the inverse problem has been solved. The thermal conductivity for temperature $T_d = 100^{\circ}$C is known, namely $\lambda_d = \lambda(100) = 379.594$ and two internal temperatures $T_{d\,1} = T\,(0.05,\ 0.05) = 345.6$ and $T_{d\,2} = T\,(0.03,\ 0.03) = 194.45$ are given. The following values of the parameters a, b, c (c.f. equation (2)) have been obtained: $a = 1.54 \cdot 10^{-5}$, $b = -0.0623$, $c = 385.673$.

Summing up, the algorithm presented allows to identify the unknown parameter as the temperature dependent function, and it is the main advantage of the approach discussed.

References

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