INCOMES PROBABILISTIC MODEL OF THE BANKING NETWORK

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Abstract. Investigation of the probabilistic model of incomes change in the banking network is presented. The closed Markov queueing network applies as model.

1. Introduction

Contemporary banking network - this is the important field of the national economy of any developed state. The pay system operational in each country can be briefly described as follows. On the upper level of banking network is located Central bank, on the second - commercial banks with their branches. According to the rules, affirmed by Central bank, under its control electronic interbank exchanges according to the pay transactions of banks and their clients are achieved. Exchanges are produced through the correspondent accounts, which are opened on the balance of each bank.

Interbank exchanges are achieved with the aid of the banking computer networks, because of which the banks increase their incomes. For decreasing of expenditures and increase in the productivity of such networks the developers increasingly more frequently turn to the mathematical models, in particular, to the models of the theory of the queuing networks (QN). The special features of the using of theory of QN require the calculation of different forms of interaction of their separate queuing systems (QS). QS interact among themselves by stage-by-stage service of applications, change in the duration of servicing in different stages etc. The similar phenomena are observed also in banking networks with various number of branches and financial settlements centers. Thus the theory of queuing networks is convenient language for the description of functioning of the banking networks representing rather difficult model for formalization. Important task is the analysis of such networks in the transient regime, which makes it possible to find the characteristics of different objects depending on the time, models of which they are.

Purpose of this work - the application of a Markov QN with the incomes for finding the expected income from the passages between the states of banking network, which correspond to the entering of requests from the commercial banks in central bank and obtaining answers on these demands. They can serve as requests for transaction for the transfer of money into another bank.

2. Analysis of the incomes of the central bank

Let each commercial bank has one correspondent account in the Central bank. By income of banking network we will understand the total sum of money, which is taking place on accounts of banks and on correspondent accounts of the Central bank which receives incomes while translating money from commercial banks with the help of a banking computer network and suffers losses while translating money into the commercial banks.

Change of the incomes in such banking network can be described with the help of the closed exponential QN, which consists of peripheral QS $S_1, S_2, \dots, S_{n-1}$ (commercial banks) and central queuing system S_n (Central bank). The state of network is described by the vector $k(t) = (k,t) = (k_1, k_2, \dots, k_{n-1}, k_n, t)$, where k_i - the count of requests in the system S_i, $i = \overline{1,n}$. By request in this case is understood the transaction on the transfer of money from the commercial or Central bank. Let μ_i - the intensity of the servicing requests in the system S_i, $i = \overline{1,n}$, p_{ni} - the probability of the entering of requests from the system S_n into the system S_i, $i = \overline{1,n-1}$, $\sum_{i=1}^{n-1} p_{ni} = 1.$

Through $v_n(k,t)$ let us designate the complete expected income, which the system S_n obtains in time t, if at the initial moment of time network is in state k (expected income of Central bank in time t). Let's assume that the system S_n earns $r_n(k)$ conventional units for the unit of time during entire period of its stay in the state k. When QN accomplishes passage from state k to state $(k - I_i + I_n)$, it brings income to QS S_n in the size $r(k - I_i + I_n)$, while when accomplishes passage from state k to state $(k + I_i - I_n)$, then brings income in the size $(-R(k + I_i - I_n))$. Let's note that $r_n(k)$, $r(k)$ and $R(k)$ have different dimensionality. During the time interval Δt the QN can remain in the state k, either complete passage into the state $(k - I_i + I_n)$ or $(k + I_i - I_n)$. If it remains in the state k, then income of QS S_n will compose $r_n(k)\Delta t$ plus the expected income $v_n(k,t)$, which it will bring for those remaining t of the units of time. The probability of this event is equal $1 - \sum_{i=1}^{n} \mu_i u(k_i)\Delta t$. On the other hand, if in time Δt QN accomplishes passage into the state $(k - I_i + I_n)$ with the probability $\mu_i u(k_i)\Delta t$, where $u(k_i) = \begin{cases} 1, k_i > 0 \\ 0, k_i = 0 \end{cases}$, then income of QS S_n will compose $r(k - I_i + I_n)$, plus the expected income $v_n(k - I_i + I_n, t)$, which will be obtained in the remaining time if state $(k - I_i + I_n)$ was the initial state; but if in time Δt QN accomplishes passage into the state

$(k+I_i-I_n)$ with the probability $\mu_n p_{ni} u(k_n)\Delta t$, then income will compose $-R(k+I_i-I_n)$, plus the expected income of QN in the remaining time if state $(k+I_i-I_n)$ was the initial state, $i=\overline{1,n-1}$. Then using a formula of total probability, we obtain:

$$v_n(k,t+\Delta t)=\left(1-\sum_{i=1}^{n}\mu_i u(k_i)\Delta t\right)\left[r_n(k)\Delta t+v_n(k,t)\right]+$$
$$+\sum_{i=1}^{n-1}\mu_i u(k_i)\Delta t\left[r(k-I_i+I_n)+v_n(k-I_i+I_n)\right]+$$
$$+\mu_n u(k_n)\Delta t\sum_{i=1}^{n-1}p_{ni}\left[-R(k+I_i-I_n)+v_n(k+I_i-I_n)\right]$$

The system of difference-differential equations for the income $v_n(k,t)$ hence is obtained

$$\frac{dv_n(k,t)}{dt}=r_n(k)+\sum_{i=1}^{n-1}\left[\mu_i u(k_i)r(k-I_i+I_n)-\mu_n u(k_n)p_{ni}R(k+I_i-I_n)\right]-$$
$$-\sum_{i=1}^{n}\mu_i u(k_i)v_n(k,t)+\sum_{i=1}^{n-1}\left[\mu_i u(k_i)v_n(k-I_i+I_n)+\mu_n u(k_n)p_{ni}v_n(k+I_i-I_n)\right]$$

In the particular case, when $r_n(k)$, $r(k)$, $R(k)$ do not depend on the states of network and are equal respectively r_n, r, R, then

$$\begin{aligned}\frac{dv_n(k,t)}{dt}&=r_n-\mu_n u(k_n)R+r\sum_{i=1}^{n-1}\mu_i u(k_i)-\sum_{i=1}^{n}\mu_i u(k_i)v_n(k,t)+\\&+\sum_{i=1}^{n-1}\left[\mu_i u(k_i)v_n(k-I_i+I_n)+\mu_n u(k_n)p_{ni}v_n(k+I_i-I_n)\right]\end{aligned}\tag{1}$$

3. Analysis of the incomes of peripheral banks and entire network

Let us examine now the incomes, which obtain peripheral systems S_i, $i=\overline{1,n-1}$. Through $v_i(k,t)$ let us designate the complete expected income, which the peripheral system S_i in time t obtains, if at the initial moment of time network is in state k, $i=\overline{1,n-1}$.

It is understandable that now already, when network accomplishes passage from the state k into the state $(k-I_i+I_n)$, it brings income to system S_i in the size $(-r(k-I_i+I_n))$, and when accomplishes passage from the state k into the state

$(k+I_i-I_n)$, then brings income in the size $R(k+I_i-I_n)$. Let's assume also that the network brings income to system S_i in the size $r_i(k)$ for the unit of time during entire period of its stay in the state k, $i=\overline{1,n-1}$. Then we will have

$$\frac{dv_i(k,t)}{dt}=r_i(k)-\sum_{j=1}^{n}\mu_j u(k_j)v_i(k,t)+$$
$$+\left[\mu_n u(k_n)p_{ni}R(k+I_i-I_n)-\mu_i u(k_i)r(k-I_i+I_n)\right]+ \quad (2)$$
$$+\sum_{j=1}^{n-1}\left[\mu_j u(k_j)v_i(k-I_j+I_n,t)+\mu_n u(k_n)p_{nj}v_i(k+I_j-I_n,t)\right],\ i=\overline{1,n-1}$$

If $r_i(k)$, $r(k)$, $R(k)$ do not depend on the states of network, then

$$\frac{dv_i(k,t)}{dt}=r_i+\left[\mu_n u(k_n)p_{ni}R-\mu_i u(k_i)r\right]-\sum_{j=1}^{n}\mu_j u(k_j)v_i(k,t)+$$
$$+\sum_{j=1}^{n-1}\left[\mu_j u(k_j)v_i(k-I_j+I_n,t)+\mu_n u(k_n)p_{nj}v_i(k+I_j-I_n,t)\right],\ i=\overline{1,n-1} \quad (3)$$

From (2) follows, that the summary income $\sum_{i=1}^{n-1}v_i(k,t)=v(k,t)$ for peripheral QS satisfies the set of equations

$$\frac{dv(k,t)}{dt}=\sum_{i=1}^{n-1}r_i(k)+\sum_{i=1}^{n-1}\left[\mu_n u(k_n)p_{ni}R(k+I_i-I_n)-\mu_i u(k_i)r(k-I_i+I_n)\right]-$$
$$-\sum_{j=1}^{n}\mu_j u(k_j)v(k,t)+\sum_{j=1}^{n-1}\left[\mu_j u(k_j)v(k-I_j+I_n,t)+\mu_n u(k_n)p_{nj}v(k+I_j-I_n,t)\right] \quad (4)$$

and the summary income $\Theta(k,t)=v(k,t)+v_n(k,t)$ of all systems of network, as it follows from (1), (4), satisfies the equations

$$\frac{d\Theta(k,t)}{dt}=\sum_{i=1}^{n}r_i(k)-\sum_{i=1}^{n}\mu_i u(k_i)\Theta(k,t)+$$
$$+\sum_{i=1}^{n-1}\left[\mu_i u(k_i)\Theta(k-I_i+I_n,t)+\mu_n u(k_n)p_{ni}\Theta(k+I_i-I_n,t)\right] \quad (5)$$

Let us note that the aggregate profit of network $\Theta(k,t)$ does not depend on $r(k)$, $R(k)$. It was to be expected this from the law of conservation of money mass in the closed network (if Central QS receives incomes, then peripheral QS suffer losses and vice versa), entire money mass in the closed network grows only due to an increase in the percentages of the money mass, which is stored in the systems of network, which is reflected by value $\sum_{i=1}^{n} r_i(k)$.

After numbering the states of QN consecutively $1,2,\ldots,l,$ set of equations for the incomes of systems S_i, $i=\overline{1,n}$, and network as a whole, as it follows from (1), (3), (5), can be presented in the matrix form

$$\frac{dV_i(t)}{dt} = Q_i + AV_i(t)$$

where $V_i^T(t) = (v_i(1,t), v_i(2,t), \ldots, v_i(l,t))$ - the vector of the incomes of system S_i at the moment of time t depending on the states of network at the initial moment of time, $i=\overline{1,n}$. For its solution it is also possible to use operating method, after assigning the vector of initial conditions $V_i(0)$. Let $U_i(s)$ - the vector of the transforms of Laplace of incomes $v_i(j,t)$, $j=\overline{1,l}$. Then

$$sU_i(s) - V_i(0) = \frac{1}{s}Q_i + AU_i(s)\text{, i.e. } (sI - A)U_i(s) = \frac{1}{s}Q_i + V_i(0)$$

Hence we obtain $U_i(s)$

$$U_i(s) = \frac{1}{s}(sI - A)^{-1}Q_i + (sI - A)^{-1}V_i(0) \tag{6}$$

The vector of incomes $V_i(t)$ can be found with the help of inverse transformation for (6). If $W(t)$, $H(t)$ - inverse transformations of Laplace of matrices $\frac{1}{s}(sI - A_i)^{-1}$ and $(sI - A)^{-1}$ respectively, then from (6) we will have

$$V_i(t) = W(t)Q_i + H(t)V_i(0)$$

4. Example

Let's examine network with the following parameters: $n = 3$, $K = 4$, $\mu_1 = 3$, $\mu_2 = 2$, $\mu_3 = 1$, $p_{n1} = p_{n2} = 0.5$, $r_n = 15$, $r = 3$, $R = 2$, $V(0) = (34,30,34,19,26,25,30,34,29,31,16,45,32,23,28)$. State vector takes the form $(k,t) = (k_1, k_2, K - k_1 - k_2)$. The states of this network will be (0,0,4), (0,1,3), (0,2,2), (0,3,1), (0,4,0), (1,0,3), (1,1,2), (1,2,1), (1,3,0), (2,0,2), (2,1,1), (2,2,0), (3,0,1), (3,1,0), (4,0,0). Let's number them respectively 1,...,15. Figure 1 presents a change of the incomes of central system $v_n(15,t)$ in the dependence on the time, and Figure 2 presents change of the total income of the network $\Theta(15,t)$.

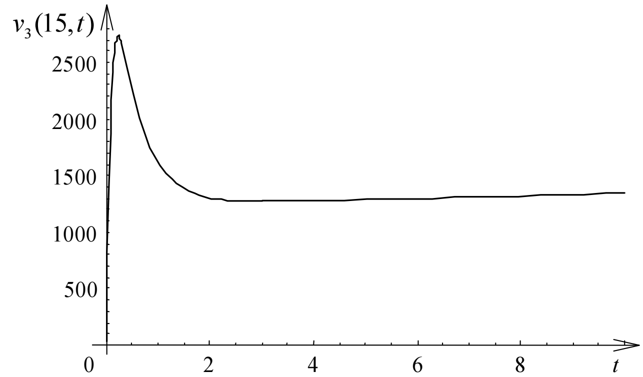

Fig. 1

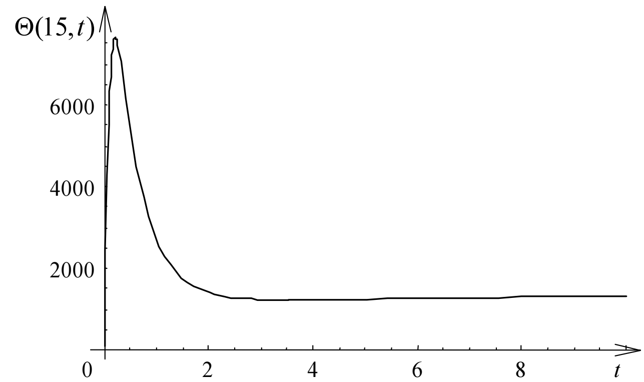

Fig. 2

Remark. The results of the calculations of different examples showed that the presence of the solutions of the set of equations of type (1), (5), (7) by operating method can be successfully conducted, when QN has relatively small number of states. This is connected, in particular, with the calculation of inverse matrices $(sI - A)^{-1}$, their decomposition and by the presence of inverse transformation of Laplace with the aid of Mathematica. Therefore it is expedient to use this method for the investigation of the models of the banking networks, which consist of the Central bank with the small number of its branches.