SOLUTION OF INVERSE HEAT CONDUCTION PROBLEM USING THE SEQUENTIAL FUNCTION SPECIFICATION METHOD

***Ewa Majchrzak*[1, 2], *Katarzyna Freus*[2], *Daniel Janisz*[1]**

[1]*Silesian University of Technology, Gliwice*
[2]*Czestochowa University of Technology, Czestochowa*

Abstract. In the paper the inverse heat conduction problem consisting in the estimation of surface heat flux from transient temperature measurements in 2D solid body is presented. In order to solve the problem the sequential function specification method using future temperature information and the boundary element method have been applied. The theoretical considerations are supplemented by the example of computations.

1. Formulation of the problem

The temperature distribution inside a heat conducting body can be described by the following boundary-initial problem

$$
\begin{cases}
x \in \Omega: & \dfrac{\partial T(x,\ t)}{\partial t} = a\nabla^2 T(x,\ t) \\
x \in \Gamma_1: & q(x,\ t) = -\lambda \dfrac{\partial T(x,\ t)}{\partial n} \\
x \in \Gamma_2: & T(x,\ t) = T_b \\
x \in \Gamma_3: & q(x,\ t) = -\lambda \dfrac{\partial T(x,\ t)}{\partial n} = q_b \\
x \in \Gamma_4: & q(x,\ t) = -\lambda \dfrac{\partial T(x,\ t)}{\partial n} = \alpha\left[T(x,\ t) - T^{\infty}\right] \\
t = 0: & T(x,\ t) = T_0
\end{cases}
\tag{1}
$$

where $x = (x_1, x_2)$, $a = \lambda/c$ is the diffusion coefficient (λ is the thermal conductivity, c is the volumetric specific heat), T_b is the boundary temperature on Γ_2, q_b is the heat flux on Γ_3, α is the heat transfer coefficient, T^{∞} is the ambient temperature, T_0 denotes the initial temperature.

For direct problems the boundary conditions as well as the initial condition, λ and c are known and we want to determine the temperature distribution $T(x, t)$.

For inverse boundary problems the boundary condition is not specified for a part or on the whole external surface of the body. Here, we assume that the heat flux $q(x, t)$ on part Γ_1 is unknown.

Additionally, some measured temperature histories are given at interior locations x_i at discrete times t^f, $f = 1, 2, \ldots, F$, this means

$$T_{d\,i}^{f} = T_d(x_1^i,\ x_2^i,\ t^f),\ \ i = 1,\ 2,\ \ldots,\ M,\ \ f = 1,\ 2,\ \ldots,\ F \tag{2}$$

2. Sequential function specification method

In sequential function specification method [1, 2] it is assumed that the heat flux is known at points $x^j \in \Gamma_1$, $j = 1, 2, \ldots, J$ at times $t^1, t^2, \ldots, t^{f-1}$, where J is the number of boundary nodes distinguished on the part Γ_1, and we want to determine the heat flux $q_j^f = q(x_1^j,\ x_2^j,\ t^f)$ at time t^f. Additionally, the temperature values are known for R future intervals, namely

$$T_{d\,i}^{f+r-1} = T_d(x_i,\ t^{f+r-1}),\ \ r = 1,\ 2,\ \ldots,\ R,\ \ i = 1,\ 2,\ \ldots,\ M \tag{3}$$

and we assume that the heat flux is constant over R future steps and equal to the heat flux at time t^f

$$q_j^f = q_j^{f+1} = \ldots = q_j^{f+R-1} \tag{4}$$

In order to solve the inverse problem, the least squares method is applied [1, 2]

$$S = \sum_{i=1}^{M} \sum_{r=1}^{R} \left(T_i^{f+r-1} - T_{d\,i}^{f+r-1} \right)^2 \tag{5}$$

Function $T_i^{f+r-1} = T(x^i,\ t^{f+r-1})$ is expanded in a Taylor series about arbitrary but known value of heat flux q_j^{*f}

$$T_i^{f+r-1} = T_i^{*f+r-1} + \sum_{j=1}^{J} \left. \frac{\partial T_i^{f+r-1}}{\partial q_j^f} \right|_{q_j^f = q_j^{*f}} \left(q_j^f - q_j^{*f} \right) \tag{6}$$

where T_i^{*f+r-1} denotes the temperature at point x^i at time t^{f+r-1} obtained under the assumption that for $t \in [t^{f-1},\ t^{f+r-1}]$ the heat flux equals $q_j^f = q_j^{f+1} = \ldots = q_j^{f+r-1} = q_j^{*f}$.

We introduce the sensitivity coefficients [1, 2] and then

$$T_i^{f+r-1} = T_i^{*f+r-1} + \sum_{j=1}^{J} Z_{j,i}^{f+r-1}\left(q_j^f - q_j^{*f}\right) \tag{7}$$

Putting (7) into (5) one has

$$S = \sum_{i=1}^{M} \sum_{r=1}^{R} \left[T_i^{*f+r-1} + \sum_{j=1}^{J} Z_{j,i}^{f+r-1}\left(q_j^f - q_j^{*f}\right) - T_{d\,i}^{f+r-1} \right]^2 \tag{8}$$

Differentiating the criterion (8) with respect to the unknown heat fluxes q_j^f and using the necessary condition of minimum, one obtains the following system of equations

$$\sum_{i=1}^{M} \sum_{r=1}^{R} \sum_{j=1}^{J} Z_{l,i}^{f+r-1} Z_{j,i}^{f+r-1}\left(q_j^f - q_j^{*f}\right) = \sum_{i=1}^{M} \sum_{r=1}^{R} Z_{l,i}^{f+r-1}\left(T_{d\,i}^{f+r-1} - T_i^{*f+r-1}\right) \tag{9}$$

where $l = 1, 2, \ldots, J$.
We introduce the rectangular matrix of dimensions $RM \times J$

$$\mathbf{Z}^f = \begin{bmatrix} Z_{1,1}^f & Z_{2,1}^f & \ldots & Z_{J,1}^f \\ \ldots & \ldots & \ldots & \ldots \\ Z_{1,1}^{f+R-1} & Z_{2,1}^{f+R-1} & \ldots & Z_{J,1}^{f+R-1} \\ Z_{1,2}^f & Z_{2,2}^f & \ldots & Z_{J,2}^f \\ \ldots & \ldots & \ldots & \ldots \\ Z_{1,2}^{f+R-1} & Z_{2,2}^{f+R-1} & \ldots & Z_{J,2}^{f+R-1} \\ Z_{1,M}^f & Z_{2,M}^f & \ldots & Z_{J,M}^f \\ \ldots & \ldots & \ldots & \ldots \\ Z_{1,M}^{f+R-1} & Z_{2,M}^{f+R-1} & \ldots & Z_{J,M}^{f+R-1} \end{bmatrix} \tag{10}$$

and then system of equations (9) can be written in the form

$$\left(\mathbf{Z}^f\right)^T \mathbf{Z}^f \mathbf{q}^f = \left(\mathbf{Z}^f\right)^T \mathbf{Z}^f \mathbf{q}^{*f} + \left(\mathbf{Z}^f\right)^T \left(\mathbf{T}_d^f - \mathbf{T}^{*f}\right) \tag{11}$$

where

$$\mathbf{q}^f = \begin{bmatrix} q_1^f \\ q_2^f \\ \dots \\ q_J^f \end{bmatrix}, \quad \mathbf{q}^{*f} = \begin{bmatrix} q_1^{*f} \\ q_2^{*f} \\ \dots \\ q_J^{*f} \end{bmatrix} \tag{12}$$

and

$$\mathbf{T}_d^f = \begin{bmatrix} T_{d1}^f & \dots & T_{d1}^{f+R-1} & T_{d2}^f & \dots & T_{d2}^{f+R-1} & T_{dM}^f & \dots & T_{dM}^{f+R-1} \end{bmatrix}^{\mathrm{T}} \tag{13}$$

while

$$\mathbf{T}_d^{*f} = \begin{bmatrix} T_{d1}^{*f} & \dots & T_{d1}^{*f+R-1} & T_{d2}^{*f} & \dots & T_{d2}^{*f+R-1} & T_{dM}^{*f} & \dots & T_{dM}^{*f+R-1} \end{bmatrix}^{\mathrm{T}} \tag{14}$$

are the matrices of dimensions $RM \times 1$.

The system of equations (10) allows to find the values of heat fluxes q_j^f at boundary nodes x^j, $j = 1, 2, \dots, J$ at time t^f.

In the sequential function specification method the sensitivity coefficients are used (c.f. matrix (10)). In order to calculate them, the governing equations (1) are differentiated with respect to the unknown heat flux $q(x^j, t)$ at boundary point $x^j \in \Gamma_1$

$$\begin{cases} x \in \Omega: & \dfrac{\partial Z_j(x,\ t)}{\partial t} = a \nabla^2 Z_j(x,\ t) \\ x \in \Gamma_1: & W_j(x,\ t) = -\lambda \dfrac{\partial Z_j(x,\ t)}{\partial n} = \begin{cases} 1, & x = x^j \\ 0, & x \neq x^j \end{cases} \\ x \in \Gamma_2: & Z_j(x,\ t) = 0 \\ x \in \Gamma_3: & W_j(x,\ t) = -\lambda \dfrac{\partial Z_j(x,\ t)}{\partial n} = 0 \\ x \in \Gamma_4: & W_j(x,\ t) = -\lambda \dfrac{\partial Z_j(x,\ t)}{\partial n} = \alpha\, Z_j(x,\ t) \\ t = 0: & Z_j(x,\ t) = 0 \end{cases} \tag{15}$$

So, in the case considered the additional boundary initial problems (15) for $j = 1, 2, \dots, J$ should be solved.

3. Boundary element method

In order to solve the direct problem (1) for arbitrary assumed value of unknown heat flux $q(x, t)$ and the additional problems (15) connected with the sensitivity coefficients, the 1st scheme of boundary element method for 2D parabolic equations has been used. So, we consider the following equation

$$x \in \Omega: \quad \frac{\partial F(x,\ t)}{\partial t} = a\nabla^2 F(x,\ t) \tag{16}$$

where $F(x, t) = T(x, t)$ for direct problem (1), while $F(x, t) = Z_j(x, t)$ for additional problems (15).

The boundary integral equation corresponding to the transition $t^{f-1} \to t^f$ takes a form [3]

$$B(\xi)F\left(\xi,\ t^f\right) + \frac{1}{c}\int_{t^{f-1}}^{t^f}\int_\Gamma F^*\left(\xi,\ x,\ t^f,\ t\right) J(x,\ t)\,\mathrm{d}\Gamma\mathrm{d}t = \\ \frac{1}{c}\int_{t^{f-1}}^{t^f}\int_\Gamma J^*\left(\xi,\ x,\ t^f,\ t\right)F(x,\ t)\,\mathrm{d}\Gamma\mathrm{d}t + \frac{1}{c}\int_\Omega F^*\left(\xi,\ x,\ t^f,\ t^{f-1}\right)F\left(x,\ t^{f-1}\right)\mathrm{d}\Omega \tag{17}$$

where $\xi = (\xi_1,\ \xi_2)$ is the observation point, $B(\xi) = 1$ for $\xi \in \Omega$ and $B(\xi) \in (0,\ 1)$ for $\xi \in \Gamma$, $F^*(\xi,\ x,\ t^f,\ t)$ is the fundamental solution

$$F^*\left(\xi,\ x,\ t^f,\ t\right) = \frac{1}{4\pi a\left(t^f - t\right)}\exp\left[-\frac{r^2}{4a\left(t^f - t\right)}\right] \tag{18}$$

where r is the distance between the points ξ and x, while $J(x,\ t) = -\lambda\partial F(x,\ t)/\partial n$ and $J^*(\xi,\ x,\ t^f,\ t) = -\lambda\partial F^*(\xi,\ x,\ t^f,\ t)/\partial n$.

For constant elements with respect to time [2, 3], the equation (17) can be written in the form

$$B(\xi)F\left(\xi,\ t^f\right) + \int_\Gamma J\left(x,\ t^f\right)g(\xi,\ x)\,\mathrm{d}\Gamma = \int_\Gamma F\left(x,\ t^f\right)h(\xi,\ x)\,\mathrm{d}\Gamma + \\ \int_\Omega F^*\left(\xi,\ x,\ t^f,\ t^{f-1}\right)F\left(x,\ t^{f-1}\right)\mathrm{d}\Omega \tag{19}$$

where

$$h\left(\xi,\ x\right)=\frac{1}{c}\int\limits_{t^{f-1}}^{t^f} J^*\left(\xi,\ x,\ t^f,\ t\right)\,\mathrm{d}t \tag{20}$$

and

$$g\left(\xi,\ x\right)=\frac{1}{c}\int\limits_{t^{f-1}}^{t^f} F^*\left(\xi,\ x,\ t^f,\ t\right)\,\mathrm{d}t \tag{21}$$

In order to solve the equation (19), the boundary Γ is divided into N constant boundary elements, the interior Ω is divided into L constant internal cells and then we obtain the following system of algebraic equations ($i = 1, 2, \ldots, N$)

$$\sum_{j=1}^{N} G_{ij} J\left(x^j,\ t^f\right)=\sum_{j=1}^{N} H_{ij} F\left(x^j,\ t^f\right)+\sum_{l=1}^{L} P_{il} F\left(x^l,\ t^{f-1}\right) \tag{22}$$

where

$$G_{ij}=\int\limits_{\Gamma_j} g\left(\xi^i,\ x\right)\mathrm{d}\Gamma_j \tag{23}$$

and

$$H_{ij}=\begin{cases}\displaystyle\int\limits_{\Gamma_j} h\left(\xi^i,\ x\right)\mathrm{d}\Gamma_j\ , & i\neq j\\ -1/2\ , & i=j\end{cases} \tag{24}$$

while

$$P_{il}=\int\limits_{\Omega_l} F^*\left(\xi^i,\ x,\ t^f,\ t^{f-1}\right)\mathrm{d}\Omega \tag{25}$$

After determining the 'missing' boundary values ($F(x^j, t^f)$ or $J(x^j, t^f)$), the values of function $F(x^i, t^f)$ at internal nodes $x^i \in \Omega$ for time t^f are calculated using the formula ($i = N+1, N+2, \ldots, N+L$)

$$F\left(x^i,\ t^f\right)=\sum_{j=1}^{N} H_{ij} F\left(x^j,\ t^f\right)-\sum_{j=1}^{N} G_{ij} J\left(x^j,\ t^f\right)+\sum_{l=1}^{L} P_{il} F\left(x^l,\ t^{f-1}\right) \tag{26}$$

4. Example of computations

The domain of dimensions 0.1×0.1 m made of copper (λ = 330 W/mK, c = = 3.7464 · 10^6 J/m^{3}K) has been considered. Initial temperature equals 1000°C. The following boundary conditions have been assumed (c.f. Figure 1)

$$\begin{array}{lll} 0 < x_1 < 0.1, & x_2 = 0.1: & q(x_1,\ 0.1,\ t) = 0 \\ x_1 = 0\ : & 0 < x_2 < 0.1: & q(0,\ x_2,\ t) = 1500\left[T(0,\ x_2,\ t) - 30\right] \\ x_1 = 0.1\ , & 0 < x_2 < 0.1: & T(0.1,\ x_2,\ t) = 500°\text{C} \end{array} \tag{27}$$

The aim of investigations is to determine the heat flux $q(x_1,\ 0,\ t)$ for $x_1 \in (0,\ 0.1)$.

As is well known, in order to solve the inverse problem, some measured temperature histories at interior locations should be given. These temperatures have been obtained from the direct problem solution under the assumption that

$$q(x_1,\ 0,\ t) = 10^5 + 2.5\ \cdot\ 10^4 t - 200t^2 \tag{28}$$

The problem has been solved using the BEM. The boundary has been divided into 20 constant boundary elements and the interior has been divided into 25 constant internal cells - Figure 1. Time step equals $\Delta t = 2$ s.

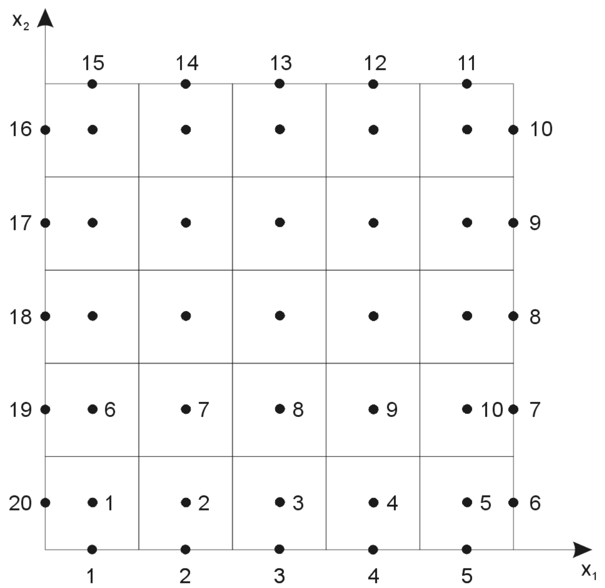

Fig. 1. Discretization

In Figure 2 the cooling curves at the internal points 1, 2, …, 10 are shown. As an example, in Figure 3 the distributions of sensitivity functions $Z_1(x,\ t)$ and $Z_2(x,\ t)$ for time 40 s are presented. Figure 4 illustrates the courses of identified heat fluxes at the boundary nodes 1, 2, 3, 4, 5 both in the case of the exact as well as the disturbed cooling curves (standard deviation: $\sigma = 1$) and for $R = 3$.

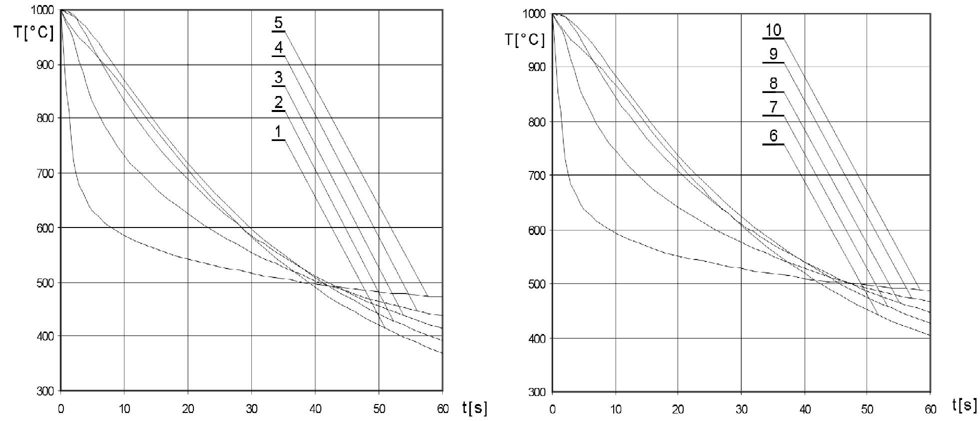

Fig. 2. Cooling curves at the internal points

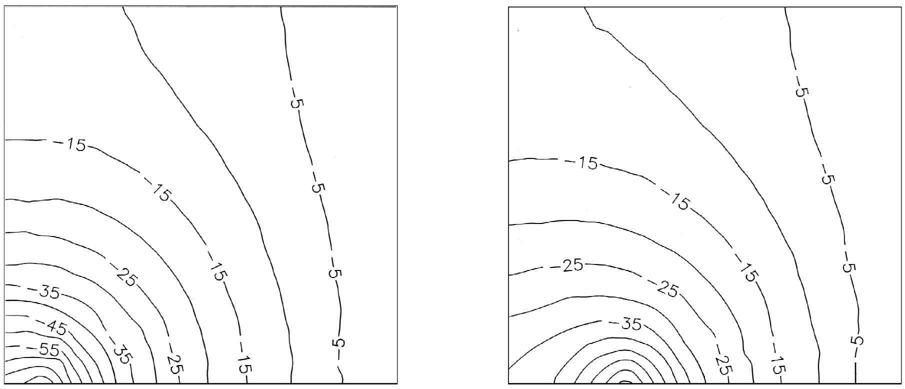

Fig. 3. Distribution of sensitivity functions $Z_1(x, 40) \cdot 10^6$

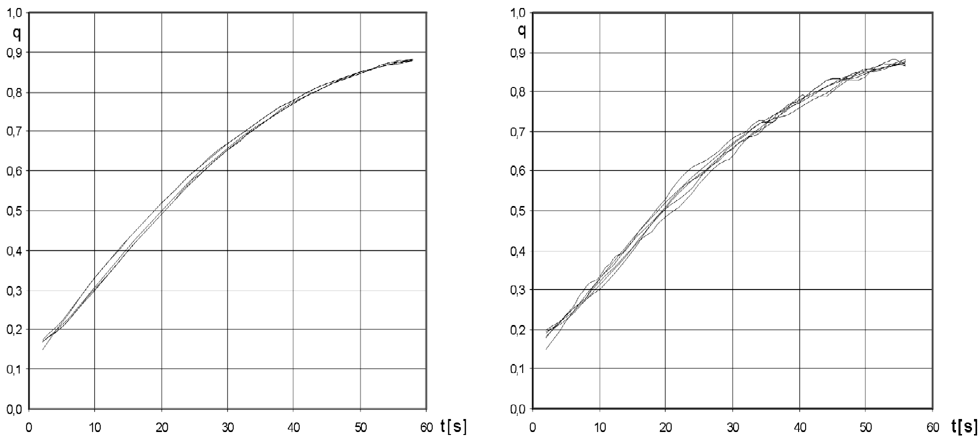

Fig. 4. Exact and identified heat fluxes [MW/m^2] at the boundary nodes - exact and disturbed data

Summing up, the sequential function specification method using future temperature measurements constitutes an effective tool of boundary heat flux identification.

The paper has been sponsored by KBN (Grant No 3 T11F 018 26).

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