A SEQUENCE OF DISCRETE ALMOST REPRESENTING MEASURES CONVERGENT TO A REPRESENTING MEASURE

Piotr Puchała

Institute of Mathematics and Computer Science, Czestochowa University of Technology

Abstract. We will construct the sequence of discrete measures almost representing points of certain compact convex sets and supported by extreme points of that sets convergent in the weak* topology to a discrete measure which represents point of a compact convex set and is supported by its extreme points.

1. Definitions

By T we will denote the metric space, by X - n-dimensional Euclidean space (although definitions and facts below can be stated in a more general setting).

1) We say that:
   a) a set $A \subset X$ is convex, if whenever it contains two points, it also contains the line segment joining them; „algebraically speaking" A is convex, if $\lambda x + (1-\lambda) y \in A$ whenever $x, y \in A$ and $0 \le \lambda \le 1$;
   b) a point $e \in A$ is an extreme point of A if and only if whenever $e = \lambda x + (1-\lambda)y,\ x, y \in A,\ 0 < \lambda < 1,$ then $x = y = e$ (by ext A we will denote the set of extreme points of A);
   c) the convex hull of $A \subset X$ (denoted by conv A) is the set of all convex combinations of points of A

$$cvA := \left\{ x : x = \sum_{i=1}^{n} \lambda_i x_i : x_i \in A,\ \lambda_i \ge 0,\ \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

2) Let $A \subset X$ be a compact convex set, $x \in A$ and $\gamma > 0$. We say that:
   a) a regular probability Borel measure μ on X represents point $x \in A$ if the equality $f(x) = \int_A f d\mu$ holds for all $f \in X^*$;
   b) a regular probability Borel measure μ on X γ- represents point $x \in A$ if the inequality $\left| f(x) - \int_A f d\mu \right| < \gamma$ holds for all $f \in X^*$;

3) Denote by $C_b(X)$ the set of all continuous bounded real functions on X. This set with the supremum norm given by the formula $\|\varphi\| := \sup\{|\varphi(x)| : x \in X\}$ is a Banach space. By $M(X)$ we will denote the space of all probability measures on the σ-algebra $B(X)$ of the Borel subsets of X. Take any such measure and consider the family of sets of the form $V_\mu(\varphi_1,...,\varphi_k,\varepsilon_1,...,\varepsilon_k):=$ $=\left\{\nu \in M(X) : \left|\int \varphi_i d\nu - \int \varphi_i d\mu\right| < \varepsilon_i, \quad i = 1,...,k\right\}$ where the functions $\varphi_i \in C_b(X)$, $\varepsilon_i > 0, i = 1,...,k$. The family of all such sets is a base of a topology on $M(X)$, called "the weak*-topology". The generalized sequence (μ_α) of measures converges to the measure μ_0 in this topology iff $\int \varphi d\mu_\alpha \to \int \varphi d\mu_0$ for any $\varphi \in C_b(X)$.
4) A multifunction P is a mapping from the space T into nonempty subsets of a space X. Let $\emptyset \neq A \subset X$. We will use the following notation:

$$P^+(A) := \{x \in X : P(x) \subseteq A\}$$
$$P^-(A) := \{x \in X : P(x) \cap A \neq \emptyset\}$$

We say that multifunction $P : T \to 2^X - \{\emptyset\}$ is:

a) lower semicontinuous, if the set $P^-(V)$ is open in T for every V open in X;

b) upper semicontinuous, if the set $P^+(V)$ is open in T for every V open in X;

c) continuous, if it is both lower- and upper semicontinuous.
5) Let P be a multifunction. A selection of P is a single-valued mapping $p : T \to X$ such that for any $x \in X$ there holds $p(x) \in P(x)$.

2. Facts

In this section we state without proof more or less known facts which will be needed in further considerations.

1) (Krein-Milman theorem) A compact convex set $A \subset X$ is equal to the convex hull of its extreme points (X - finite dimensional).
2) A multifunction $P : T \to 2^X - \{\emptyset\}$ is lower semicontinuous if and only if for every sequence $(t_n) \subset T$ and any point $x_0 \in P(t_0)$ there exists sequence $(x_n) \subset X$ convergent to x_0 and such that $x_n \in P(t_n)$.
3) (Michael selection theorem) Any lower semicontinuous multifunction from a paracompact space into space of nonempty subsets of a Banach space with closed convex values has a continuous selection.

3. Construction

Let T be a metric space, $X - n$ dimensional Euclidean space, $P: T \to 2^X - \{\varnothing\}$ - continuous multifunction with compact convex values. In this case multifunction

$$t \to \operatorname{ext} P(t)$$

is lower semicontinuous (see [3]).
Choose and fix $\gamma > 0$ and a continuous selection $p(\cdot)$ of $P(\cdot)$.

Let (t_n) be a sequence in T, convergent to the point $(t_0) \in T$. Consider point $p(t_0) \in P(t_0)$. By the Krein-Milman theorem there exist points $a_1, \ldots, a_m \in extP(t_0)$, positive numbers $\lambda_1, \ldots, \lambda_m, \sum_{i=1}^{m} \lambda_i = 1$, such that $p(t_0) = \sum_{i=1}^{m} \lambda_i a_i$. Then we can check that the discrete measure (i.e. the measure being a convex combination of Dirac measures) $\mu_0 := \sum_{i=1}^{m} \lambda_i \delta_{a_i}$ represents point $p(t_0)$. As multifunction $extP(\cdot)$ is lower semicontinuous, then for any a_i there exists sequence $b_n^i \in \operatorname{ext} P(t_n)$ convergent to a_i. Define measure

$$\mu_n := \sum_{i=1}^{m} \lambda_i \delta_{b_n^i}$$

and let $\varphi \in C_b(X)$. We then have

$$\left| \int \varphi d\mu_n - \int \varphi d\mu_0 \right| \le \sum_{i=1}^{m} \lambda_i \left| \varphi\left(b_n^i\right) - \varphi\left(a_i\right) \right| \xrightarrow[n \to \infty]{} 0$$

This proves that the sequence (μ_n) converges weakly* to the measure (μ_0).
Moreover, for any $f \in X^*$ we have

$$\left| f(p(t_n)) - \int_{P(t_n)} f d\mu_n \right| \le \left| f(p(t_n)) - f(p(t_0)) \right| + \left| f(p(t_0)) - \int_{P(t_0)} f d\mu_0 \right| + $$
$$+ \left| \int_{P(t_0)} f d\mu_0 - \int_{P(t_n)} f d\mu_n \right|$$

The term on the right converges to 0, the second equals 0 because measure μ_0 represents point $p(t_0)$. For the third term there holds

$$\left| \int\limits_{P(t_0)} f d\mu_0 - \int\limits_{P(t_n)} f d\mu_n \right| \le \sum_{i=1}^{m} \lambda_i \left| f\left(a_i - b_n^i\right) \right| \xrightarrow[n\to\infty]{} 0$$

Hence there exists natural number n_0 such that for each $n \geq n_0$ the measure μ_n γ-represents point $p(t_n)$. Finally, by construction, $\mu_n\left(extP(t_n)\right) = 1$.

References

[1] Aliprantis Ch.D., Border K.C., Infinite dimensional analysis. A hitchhiker's guide, 2nd edition, Springer Verlag 1999.

[2] Phelps R.R., Lectures on Choquet theorem, 2nd edition, Springer Lecture Notes in Mathematics 1757, 2000.

[3] Puchała P., On a lower semicontinuity of a certain multifunction, Scientific Research of the Institute of Mathematics and Computer Science 2003, 1, 2, 153-158.

[4] Webster R., Convexity, Oxford University Press 1994.