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A CONTRIBUTION TO THE MACROSCOPIC MODELLING OF HYPERBOLIC HEAT TRANSFER IN A MICRO-PERIODIC RIGID CONDUCTOR

Urszula Siedlecka

Institute of Mathematics and Computer Science, Czestochowa University of Technology

Abstract. The aim of this contribution is to analyze the propagation of initial temperature disturbances on a macroscopic level. The main result of in this contribution is a formulation of an independent system of equations describing the above problem. The derived macroscopic heat transfer equations will be represented by the independent temperature disturbance equations and by the equation for the microscopic (averaged) temperature field. To make this note self-consistent we outline in the subsequent sections the basic preliminary concepts of the tolerance averaging technique. The detailed discussion of this technique can be found in [5].

1. Object of the paper

The object of considerations is a heat propagation in the micro-periodic rigid conductor based on the hyperbolic heat conduction law. Problems of this kind were investigated in a series of papers and can be described by using various heat propagation constitutive laws [2-4]. In this contribution we restrict considerations to the simplest hyperbolic heat transfer law which is due to Cattaneo and takes into account only one relaxation time t. The main attention will be devoted to rigid conductors with a micro-periodic non-homogenous structure. Thus, the governing hyperbolic heat transfer equation has highly oscillating and non-continuous functional coefficients. That is why we shall deal with a certain macroscopic model of the problem under consideration which will be based on the tolerance averaging technique which has been summarized in [5] and developed in a series of papers.

Notation: Superscripts A, B ran over the sequence $1,2,\ldots,N$, summation convention holds. Gradients with respect to the space coordinates will be denoted by ∇ and time derivative by overdot.

2. Preliminaries

We shall deal with a rigid heat conductor which has Δ-periodic non-homogenous structure, with $\Delta = (-l_1/2,l_1/2)\times(-l_2/2,l_2/2)\times(-l_3/2,l_3/2)$. It is assumed that diameter l of a periodic cell Δ is sufficiently small when compared to the minimum characteristic length dimension of a region Ω in the 3-space R^3 occupied by the conductor. Setting $\mathbf{x} = (x_1, x_2, x_3)$ we define $\Delta(\mathbf{x}) = \mathbf{x} + \Delta$ as a periodicity cell with a center at a point $\mathbf{x}$. We also define $\Omega_0 := \{\mathbf{x} \in \Omega : \Delta(\mathbf{x}) \subset \Omega\}$ and introduce averaging operator setting

$$\langle f \rangle(\mathbf{x}) = \frac{1}{|\Delta|} \int\limits_{\Delta(\mathbf{x})} f(\mathbf{z})\, d\mathbf{z}, \quad \mathbf{x} \in \Omega_0 \tag{1}$$

where f is an arbitrary integrable function. We recall that function $F(\cdot)$ defined in Ω is slowly-varying (with respect to a certain tolerance system T [5]) $F \in SV_\Delta(T)$, if the following tolerance approximation formula

$$\langle gF \rangle(\mathbf{x}) \cong \langle g \rangle(\mathbf{x}) F(\mathbf{x}), \quad \mathbf{x} \in \Omega_0 \tag{2}$$

holds for an arbitrary integrable function g defined in Ω. Let $\theta(\mathbf{x},t)$, $\mathbf{x} \in \Omega$ is a temperature field at time t. Define $\vartheta(\mathbf{x},t) \equiv \langle \theta \rangle(\mathbf{x},t)$ as a macroscopic temperature.

The first fundamental restriction imposed on the temperature field in the framework of the tolerance averaging technique will be given by the following decomposition of the temperature field

$$\theta(\mathbf{x},t) = \vartheta(\mathbf{x},t) + \psi(\mathbf{x},t) \tag{3}$$

together with the assumption

$$\vartheta(\cdot,t) \in SV_\Delta(T) \tag{4}$$

Thus, we conclude that

$$\langle \psi \rangle(\mathbf{x},t) \cong 0 \tag{5}$$

and hence ψ can be interpreted as a certain disturbance of a temperature field θ.

The second fundamental restriction of the tolerance averaging technique is imposed on the temperature disturbance ψ and is given by

$$\psi(\mathbf{x},t) = h^A(\mathbf{x}) \vartheta^A(\mathbf{x},t), \quad \mathbf{x} \in \Omega_0 \tag{6}$$

where $h^A(\cdot)$, $A = 1,...,N$ is a system of Δ-periodic continuous functions which are piecewise smooth and satisfy condition $\langle h^A \rangle = 0$. Moreover, values of h^A are of an order $O(l)$. Functions h^A, termed shape functions, are assumed to be known *a priori* in every special problem under consideration. The form of shape functions can be obtained by a periodic discretization of a cell Δ, cf. [6]. Moreover, we assume that new basic unknowns $\vartheta^A(\mathbf{x},t)$, $\mathbf{x} \in \Omega_0$, are slowly-varying functions

$$\vartheta^A(\cdot,t) \in SV_\Delta(T) \tag{7}$$

and will be called *temperature fluctuations amplitudes*.

3. Macroscopic model equations

The starting point of the modeling procedure is a known Cattaneo hyperbolic heat transfer equation with one relaxation time

$$c_\tau \ddot{\theta} + c\dot{\theta} - \nabla \cdot (\mathbf{A} \cdot \nabla \theta) = 0$$

where $\mathbf{A}$, c, c_τ are: a heat conduction tensor, a specific heat and the modified specific heat with relaxation time τ. For a non-homogenous micro-periodic conductor c, c_τ and $\mathbf{A}$ are Δ-periodic functions of $\mathbf{x}$ which, as a rule are non-continuous and highly-oscillating.

The tolerance averaging technique applied to the above equation leads to the following system of equations with constant coefficients for a micro-temperature ϑ and temperature fluctuation ϑ^A

$$\begin{cases} \langle c_\tau \rangle \ddot{\vartheta} + \langle c \rangle \dot{\vartheta} - \nabla(\langle \mathbf{A} \rangle \cdot \nabla \vartheta + \langle \mathbf{A} h^A \rangle \cdot \nabla \vartheta^A) = 0 \\ \langle c_\tau h^A h^B \rangle \ddot{\vartheta}^B + \langle c h^A h^B \rangle \dot{\vartheta}^B + \langle \nabla h^A \cdot \mathbf{A} \cdot \nabla h^B \rangle \cdot \vartheta^B + \langle \nabla h^A \cdot \mathbf{A} \rangle \cdot \nabla \vartheta = 0, \ A = 1, \ldots N \end{cases} \tag{8}$$

It has to be remembered that the above equations have a physical sense if $\vartheta(\cdot,t)$, $\vartheta^A(\cdot,t)$ are slowly-varying functions according to formulae (4) and (7). Equations (8) have to be satisfied for every $\mathbf{x} \in \Omega$. Due to the fact that equations (8) for ϑ^A are ordinary deferential equations we have to formulate the boundary conditions only for the macroscopic temperature field ϑ, provided that the region Ω is bounded. Subsequently we shall deal with initial value problems for (8) by asumming that $t \geq 0$. On a macroscopic level initial conditions for a temperature field θ are given by

$$\theta(\mathbf{x},0) = \overline{\theta}(\mathbf{x}), \ \dot{\theta}(\mathbf{x},0) = \tilde{\theta}(\mathbf{x}), \mathbf{x} \in \Omega \tag{9}$$

By means of the composition (3) and assumption (6) for a microscopic temperature ϑ and temperature fluctuation ϑ^A we obtain

$$\begin{cases} \vartheta(\mathbf{x},0) + h^A(\mathbf{x})\vartheta^A(\mathbf{x},0) = \overline{\theta}(\mathbf{x}) \\ \dot{\vartheta}(\mathbf{x},0) + h^A(\mathbf{x})\dot{\vartheta}^A(\mathbf{x},0) = \tilde{\theta}(\mathbf{x}) \end{cases}, \ \mathbf{x} \in \Omega_0 \tag{10}$$

Bearing in mind conditions (10) we shall restrict considerations to the class of initial conditions (9) given by

$$\begin{cases} \overline{\theta}(\mathbf{x}) = \overline{\vartheta}(\mathbf{x}) + h^A(\mathbf{x})\overline{\vartheta}^A(\mathbf{x}) \\ \tilde{\theta}(\mathbf{x}) = \tilde{\vartheta}(\mathbf{x}) + h^A(\mathbf{x})\tilde{\vartheta}^A(\mathbf{x}) \end{cases}, \ \mathbf{x} \in \Omega_0 \tag{11}$$

Under the aforementioned restriction the initial conditions for equations (8) take the form

$$\begin{cases} \vartheta(\mathbf{x},0)=\overline{\vartheta}(\mathbf{x}), \;\; \dot{\vartheta}(\mathbf{x},0)=\tilde{\vartheta}(\mathbf{x}) \\ \vartheta^A(\mathbf{x},0)=\overline{\vartheta}^A(\mathbf{x}), \;\; \dot{\vartheta}^A(\mathbf{x},0)=\tilde{\vartheta}^A(\mathbf{x}) \end{cases}, \;\; \mathbf{x}\in\Omega_\mathbf{0} \tag{12}$$

where $\overline{\vartheta},\tilde{\vartheta},\overline{\vartheta}^A,\tilde{\vartheta}^A$ are assumed to be known.

4. Analysis

In the subsequent analysis considerations will be restricted to the initial value problems for equations (8); we tacitly assume that the boundary condition for the macroscopic temperature field ϑ is satisfied. The aim of the analysis is to answer the question how an initial value on the temperature fluctuations $\overline{\vartheta}^A,\tilde{\vartheta}^A$ propagates for $t > 0$. To this end we shall decompose the temperature fluctuations ϑ^A into three terms

$$\vartheta^A=\vartheta_0^A+\vartheta_1^A+\vartheta_2^A \tag{13}$$

Fields ϑ_0^A have to satisfy equation

$$\langle\nabla h^A\cdot\mathbf{A}\cdot\nabla h^B\rangle\vartheta_0^B=-\langle\nabla h^A\cdot\mathbf{A}\rangle\cdot\nabla\vartheta \tag{14}$$

and ϑ_1^A are governed by

$$\langle c_\tau h^A h^B\rangle\ddot{\vartheta}_1^B+\langle c h^A h^B\rangle\dot{\vartheta}_1^B+\langle\nabla h^A\cdot\mathbf{A}\cdot\nabla h^B\rangle\vartheta_1^B=0 \tag{15}$$

together with initial conditions

$$\begin{cases} \vartheta_1^A(\mathbf{x},0)=\overline{\vartheta}^A(\mathbf{x}) \\ \dot{\vartheta}_1^A(\mathbf{x},0)=\tilde{\vartheta}^A(\mathbf{x}) \end{cases}, \;\; A=1,\ldots,N, \;\; \mathbf{x}\in\Omega_\mathbf{0} \tag{16}$$

Hence fields ϑ_2^A have to satisfy the following system of equations:

$$\langle c_\tau h^A h^B\rangle\ddot{\vartheta}_2^B+\langle c h^A h^B\rangle\dot{\vartheta}_2^B+\langle\nabla h^A\cdot\mathbf{A}\cdot\nabla h^B\rangle\vartheta_2^B=-\langle c_\tau h^A h^B\rangle\ddot{\vartheta}_0^B-\langle c h^A h^B\rangle\dot{\vartheta}_0^B \tag{17}$$

together with homogenous initial conditions

$$\begin{cases} \vartheta_2^A\left(\mathbf{x},0\right)=0 \\ \dot{\vartheta}_2^A\left(\mathbf{x},0\right)=0 \end{cases}, \;\; A=1,\ldots,N, \;\; \mathbf{x}\in\Omega_\mathbf{0} \tag{18}$$

Let us observe that $\langle c_\tau h^A h^B\rangle \in O(l^2)$ and $\langle c h^A h^B\rangle \in O(l^2)$. Now we shall introduce an additional approximation that equation (17) has to be satisfied after neglecting terms of an order $O(l^2)$ situated on the right-hand side of (17). Under this approximation, bearing in mind conditions (18), we conclude that $\vartheta_2^B \cong 0$ and hence the decomposition (13) can be approximated by

$$\vartheta^A \cong \vartheta_0^A + \vartheta_1^A,\ A = 1,...N \tag{19}$$

The final conclusion is that equation (8) can be approximated by a system of equations (14) and (15). At the same time functions ϑ_1^A have to satisfy initial conditions (16).

Let us observe that the first from equations (8) combined with (14) leads to the following equation for the microscopic temperature θ

$$\langle c_\tau\rangle\ddot{\vartheta} + \langle c\rangle\dot{\vartheta} - \mathbf{A}^0 : \nabla\nabla\vartheta = \langle \mathbf{A}\cdot\nabla h^A\rangle\cdot\nabla\vartheta_1^A \tag{20}$$

Here $\mathbf{A}^0$ is defined by $\mathbf{A}^0 = \langle\mathbf{A}\rangle - \langle\mathbf{A}\cdot\nabla h^A\rangle \mathrm{K}^{AB}\langle\nabla h^B\cdot\mathbf{A}\rangle$ where K^{AB} represents the linear transformation $R^N \to R^N$ given by $\mathrm{K}^{AB} = [\langle\nabla h^A\cdot\mathbf{A}\cdot\nabla h^B\rangle]^{-1}$.

Eqs (20) together with initial and boundary conditions for the macroscopic temperature ϑ and eqs (15) together with initial conditions (16) for ϑ_1^A constitute the final macroscopic model for the analysis of initial-boundary value problems. After obtaining solution $\vartheta, \vartheta^A, A = 1,...,N$ to a certain problem we obtain from (3), (6), (14) and (19) the formula

$$\theta(\mathbf{x},t) = \vartheta(\mathbf{x},t) - h^A(\mathbf{x})\mathrm{K}^{AB}\langle\nabla h^B\cdot\mathbf{A}\rangle\cdot\nabla\vartheta(\mathbf{x},t) + \nabla h^A\vartheta_1^A(\mathbf{x},t) \tag{21}$$

which have to hold for every $\mathbf{x} \in \Omega_0$ and $t \geq 0$ and which describes the distribution of the temperature field in a periodic conductor under consideration.

5. Conclusions

The main conclusion of this contribution is that the temperature field θ in a periodically non-homogenous rigid conductor is determined by a part depending on the macroscopic temperature field ϑ and a certain fields ϑ_1^A governed by eqs (15) and initial conditions (16). The second conclusion is that the disturbances of initial temperature, described by $\bar{\vartheta}^A(\mathbf{x}),\ \tilde{\vartheta}^A(\mathbf{x}),\ \mathbf{x}\in\Omega_0$, propagate according to the ordinary deferential equations (15). It has to be emphasized that the problem under consideration cannot be solved in the framework of the well known homogenous model of a periodic conductor, since this model does not involve coefficients depending on the period lengths [1]. The discussion of the propagation problem for the above initial temperature disturbances will be given during the presentation of this contribution.

Example of applications of the obtaied model equation (15) will be explained in widened version of this note.

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