BOUNDARY AND INITIAL FLUCTUATION EFFECT ON DYNAMIC BEHAVIOUR OF A LAMINATED SOLID

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Abstract. In the paper an approximate description of the evolution of boundary traction fluctuations in the two-component linear-elastic laminates is discussed. In the paper [1] it has been shown that in the framework of the tolerance averaged model of linear elastodynamics of the aforementioned laminates, cf. [2], in the first approximation such evolution is independent of the mean (averaged) displacement. The aim of this contribution is to discuss solutions to the boundary problem formulated in [1].

Introduction

The subject of the paper is to discuss the boundary effects in elastodynamics of periodic two-laminated linear-elastic medium. The approximate description of elastodynamics of linear-elastic two-laminated periodic medium explained in [1] will be applied to the analysis and discussion behaviours which can be treated as a certain consequence of boundary traction fluctuations. This description was realized in the framework of the simplified tolerance averaged model of the laminated periodic medium being an adaptation of the classical tolerance averaged model. However, in this adaptation an additional approximate assumption was taken into account. Basic unknowns of this simplified model are taken: the averaged displacement field and the field of the intrinsic displacement fluctuations. The basic property of the applied model is that the effective modulae tensor plays an important role in equations of this model. An example of the laminated medium under consideration is shown in Figure1 but we shall also deal with the laminated semi-space which can be obtained by the limit passage $L\to\infty$, $H_2\to\infty$ and $H_3\to\infty$. The aim of this note is to discuss a certain boundary problem for the periodically laminated two-constituent medium in which the part of the boundary perpendicular to the direction of the laminae is subjected to the oscillating tractions.

Denotations

Points of the physical space are denoted by $\mathbf{x} = (x_1,x_2,x_3)$ and t stands for a time coordinate. Partial differentiation with respect to the x_k-coordinate will be denoted by ∂_k, k = 1,2,3, and time differentiation by the overdot. We also introduce operators $\nabla = (\partial_1,\partial_2,\partial_3)$ and $\overline{\nabla} = (0,\partial_2,\partial_3)$. For scalar and double-scalar products we use dot and double dot, respectively. The averaging operator defined for an arbitrary integrable function f defined in (–L,L) by

$$\langle f\rangle(x) = \frac{1}{l}\int_{x-l/2}^{x+l/2} f(y)dy, \quad x \in (-L+l/2, L-l/2)$$

where $x = x_1$ will be also applied. Function f can also depends on arguments x_2, x_3 and t as parameters. Obviously, $\langle f\rangle$ is independent on x_1 provided that f is l-periodic function.

1. Model equations

Considerations will be carried out in the Cartesian orthogonal coordinate system $Ox_1x_2x_3$ where the periodic structure of a laminated medium takes place in the Ox_1-axis direction with the period l. We shall assume that $l \ll L$.

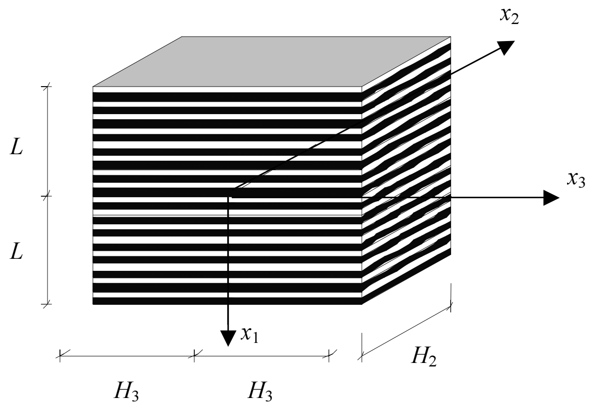

Fig. 1. Example of a laminated solid

The starting point of the considerations is the averaged 3D-model of a linear-elastic laminated medium obtained in [1] by the applying the tolerance averaging technique to the well-known equation of linear elastodynamics for laminated medium with microperiodic structure which will be rewritten in the form

$$\rho\ddot{\mathbf{U}} - \nabla\cdot(\mathbb{C} : \nabla\mathbf{U}) = \mathbf{0} \tag{1}$$

where $\rho(\cdot)$ and $\mathbb{C}(\cdot)$ are l-periodic piecewise constant functions of an argument x_1 and $\mathbf{U}$ is a displacement field. This technique is based on the concept of slowly--varying function and leads to a certain approximation of the equation (1) as well as on some restrictions imposed on the class of fields representing basic unknowns in the problem under consideration; for particulars the reader is referred to [2].

In the framework of tolerance averaged model obtained in [1] the kinematical state is described by the two basic unknowns **u** and **w** which will be referred to as the *macrodisplacements* and *intrinsic fluctuations*, respectively. Equations of this model will be rewritten in the form

$$\begin{aligned} &\langle\rho\rangle\ddot{\mathbf{u}} - \nabla\cdot(\mathbb{C}^{o}:\nabla\mathbf{u}) = -\mathbb{M}:\nabla\mathbf{w} \\ &l^{2}\langle\rho\rangle\ddot{\mathbf{w}} - l^{2}\nabla\cdot(\langle\mathbb{C}\rangle:\bar{\nabla}\mathbf{w}) + \mathbb{N}\cdot\mathbf{w} = \mathbf{0} \end{aligned} \tag{2}$$

which have a physical sense provided that basic unknowns **u** and **w** together with all partial derivatives of these fields taken into account in the model are slowly-varying functions of x_1, cf. [2]. To specify the form of the model equations (2) in every special problem the function $h(\cdot)$, referred to as a *shape function*, have to be specified. Every shape function have to satisfy conditions

$$\langle\rho h\rangle = 0,\ h\in O(l),\ \partial h\in O(l) \tag{3}$$

Matrix coefficients $\mathbb{M}=(M_{ijk})$, $\mathbb{N}=(M_{ij})$ depend on the function $h(\cdot)$ by the formulas

$$\begin{aligned} &\mathbb{M} = \langle\mathbb{C}\cdot\nabla h\rangle, \quad \mathbb{M}^{T} = \langle\nabla h\cdot\mathbb{C}\rangle \\ &\mathbb{N} = \langle\nabla h\cdot\mathbb{C}\cdot\nabla h\rangle \end{aligned} \tag{4}$$

For the two-constituent linear-elastic laminated medium with microperiodic structure as a shape function $h(\cdot)$ usually the saw-lake function, presented in Figure 2, is taken into account. In this case let the mass densities and tensors of elastic moduli for components at the laminate will be denoted by ρ', ρ'' and $\mathbb{C}'$, $\mathbb{C}''$, respectively. Let us also define $v' = l'/l$, $v'' = l''/l$. Under these denotations the coefficients $\mathbb{M}=(M_{ijk})$, $\mathbb{N}=(M_{ij})$ represented by the formulas (4) in the tolerance model equations will be given by

$$\begin{aligned} &\langle\rho\rangle = v'\rho' + v''\rho'', \qquad \langle\mathbb{C}\rangle = v'\mathbb{C}' + v''\mathbb{C}'' \\ &\mathbb{M} = \langle\mathbb{C}\cdot\nabla h\rangle = 2\sqrt{3}(\mathbb{C}' - \mathbb{C}'')\cdot\mathbf{e}_1, \quad \mathbb{M}^{T} = \langle\nabla h\cdot\mathbb{C}\rangle = 2\sqrt{3}\mathbf{e}_1\cdot(\mathbb{C}' - \mathbb{C}'') \\ &\mathbb{N} = \langle\nabla h\cdot\mathbb{C}\cdot\nabla h\rangle = \frac{12}{v'v''}\mathbf{e}_1\cdot(v''\mathbb{C}' + v'\mathbb{C}'')\cdot\mathbf{e}_1 \end{aligned}$$

where $\mathbf{e}_1$ is the unit vector of the x_1-axis. Bearing in mind that $\langle\rho h\rangle = 0$ basic unknowns $\mathbf{u}(\cdot)$, $\mathbf{v}(\cdot)$ have to be satisfy conditions $\mathbf{u} = \langle\mathbf{U}\rangle$ and $\langle\mathbf{v}\rangle = \mathbf{U}$ and should be interpreted as averaged displacement field and oscillating field, respectively. Thus the kinematics of the two-component laminated medium in the framework of the mentioned above model will be determined by macroscopic displacements **u** and fluctuation amplitudes **v**.

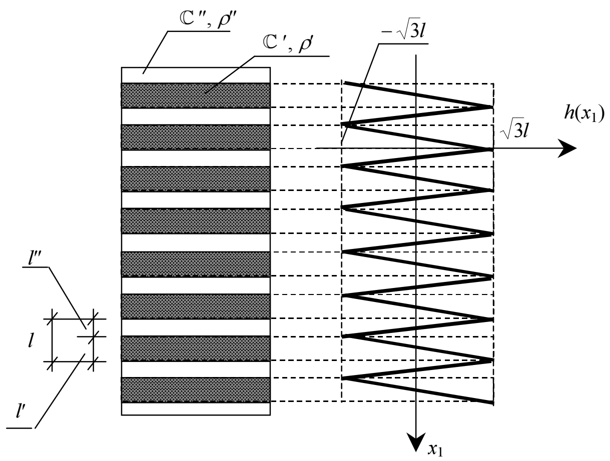

Fig. 2. Fragment of the laminated 2-component medium and the diagram of the shape function h

In order to formulate and investigate initial-boundary value problems in dynamics of the laminated solid equations (2) have to be considered together with pertinent boundary and initial conditions. We shall assume that the equations (2) for every $t > 0$ are satisfied in region $(-L, L)\times\Omega$, where Ω is a regular region in the Ox_2x_3-plane. For the laminated prism shown in Figure 1, we have $\Omega = (0,H_2)\times(-H_3,H_3)$. Moreover, L is assumed to be sufficiently large when compared to the period length l. In this case equations (2) should be considered together with the initial conditions for $t = 0$ and boundary conditions on $(-L,L)\times\partial\Omega$ as well as on $\{-L\}\times\Omega$, $\{L\}\times\Omega$. In this contribution we shall study some special boundary and initial-value problems which describe the effect of boundary traction fluctuations and initial displacement fluctuations, respectively, on the dynamic behaviour of the medium. Bearing in mind results obtained in [1] the exact displacement field can be determined from the approximation formula

$$\mathbf{U}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) - h(x_1)\mathbf{K}\cdot(\langle\nabla h\cdot\mathbb{C}\rangle : \nabla\mathbf{u}(\mathbf{x},t)) + h(x_1)\mathbf{w}(\mathbf{x},t) + O(l^3) \tag{5}$$

The main feature of equations (2) is that for function $\mathbf{w}$ we have obtained equation which is independent of $\mathbf{u}$. Let us observe that non-trivial solutions for $\mathbf{w}$ can be obtained only if boundary and initial conditions imposed on $\mathbf{w}$ are non-homogeneous. That is why the second from equations (2) characterizes the initial and boundary effects related to the fluctuation part $h\mathbf{w}$ of a displacement field $\mathbf{u}$**.** Function $\mathbf{w}$ will be referred to as fluctuation variable. Equations (2) represent a new simplified model of the laminated medium and constitute the starting point for the analysis of some initial-boundary value problems which will be carried out in the subsequent sections.

2. Initial-boundary value problem

To make this paper self-consistent, bearing in mind results obtained in [1], we are to formulate the boundary value problem which will be analyzed in this paper. In order to formulate the initial and boundary conditions for equations (2) let us rewrite formula (5) by neglecting term $O(l^3)$

$$\mathbf{U}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) - h(x_1)\mathbf{K}\cdot(\langle\nabla h\cdot\mathbb{C}\rangle : \nabla\mathbf{u}(\mathbf{x},t)) + h(x_1)\mathbf{w}(\mathbf{x},t) \tag{6}$$

Formula (6) has to hold for every $\mathbf{x} \in \langle -L,L\rangle \times \bar{\Omega}$ and every $t \geq 0$. In order to formulate traction boundary conditions we have to determine the stress tensor $\mathbf{T}$ in term of $\mathbf{u}$ and $\mathbf{w}$. Bearing in mind that $\mathbf{T}=\mathbb{C}:\nabla\mathbf{u}$, taking into account formula (6), after simple calculations and denotation $\bar{\mathbb{C}} = \mathbb{C} - (\mathbb{C}\cdot\nabla h)\cdot\mathbf{K}\cdot\langle\nabla h\cdot\mathbb{C}\rangle$ we obtain

$$\mathbf{T}(\mathbf{x},t) = \bar{\mathbb{C}} : \nabla\mathbf{u}(\mathbf{x},t) + \langle\mathbb{C}\cdot\nabla h\rangle\cdot\mathbf{w}(\mathbf{x},t) + h(x_1)\mathbb{C} : \bar{\nabla}\mathbf{v}(\mathbf{x},t) \tag{7}$$

where we recall that

$$\mathbf{v}(\mathbf{x},t) = -\mathbf{K}\cdot(\langle\nabla h\cdot\mathbb{C}\rangle : \nabla\mathbf{u}(\mathbf{x},t)) + \mathbf{w}(\mathbf{x},t) \tag{8}$$

Using (6) we have to restrict initial conditions to the form:

$$\begin{aligned} &\mathbf{u}(\mathbf{x},0) = \mathbf{u}_o(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{u}_1(\mathbf{x}) \\ &\mathbf{w}(\mathbf{x},0) = \mathbf{w}_o(\mathbf{x}), \quad \dot{\mathbf{w}}(\mathbf{x},0) = \mathbf{w}_1(\mathbf{x}) \end{aligned} \tag{9}$$

where $\mathbf{u}_o$, $\mathbf{u}_1$, $\mathbf{w}_o$ and $\mathbf{w}_1$ are assumed to be the known slowly-varying functions of x_1. Under assumption that $h(L) = h(-L) = 0$, we introduce the displacement boundary conditions on $\{L\}\times\Omega$ and $\{-L\}\times\Omega$ for every $t \geq 0$ given respectively by:

$$\begin{aligned} &\mathbf{u}(L,x_2,x_3,t) = \mathbf{u}_+(x_2,x_3,t) \\ &\mathbf{u}(-L,x_2,x_3,t) = \mathbf{u}_-(x_2,x_3,t), \qquad (x_2,x_3)\in\Omega \end{aligned} \tag{10}$$

where $\mathbf{u}_+$ and $\mathbf{u}_-$ are known. Moreover, we shall postulate the traction boundary conditions on $(-L,L)\times\partial\Omega$. Let $\mathbf{R}$ be the diagonal 3×3 matrix, $\mathbf{R} = \mathrm{diag}\{R_{11},R_{22},R_{33}\}$, which is l-periodic function of x_1 and has to satisfy conditions $\langle\mathbf{R}\rangle = \mathbf{0}$, $det\langle\mathbf{R}^2\rangle > 0$. The class of boundary tractions $\mathbf{t}$ which are assumed to be applied on $(-L,L)\times\partial\Omega$ will be restricted to the form $\bar{\mathbb{C}} = \mathbb{C} - (\mathbb{C}\cdot\nabla h)\cdot\mathbf{K}\cdot\langle\nabla h\cdot\mathbb{C}\rangle$ where $x_1 \in (-L,L)$ and for every $\bar{\mathbf{x}} \equiv (x_2,x_3) \in \partial\Omega$, $t \geq 0$ functions $\mathbf{p}(\cdot,\bar{\mathbf{x}},t)$ and $\mathbf{q}(\cdot,\bar{\mathbf{x}},t)$ are assumed to be the known slowly varying functions. In many cases $R_{11} = R_{22} = R_{33} = \partial_1 h(x_1)$; it means that the oscillating part $\mathbf{Rq}$ of boundary tractions $\mathbf{t}$ is piecewise constant in direction of the x_1-axis. From the above decomposition of tractions $\mathbf{t}$ we obtain $\langle\mathbf{t}\rangle = \mathbf{p}$, $\langle\mathbf{R}^2\rangle^{-1}\cdot\langle\mathbf{Rt}\rangle = \mathbf{q}$. Let $\mathbf{n}$ be a unit normal to $\partial\Omega$, which is outward to Ω. Substituting $\mathbf{t} = \mathbf{T}\cdot\mathbf{n}$ into the above formulae and taking into account (13) we obtain the following form of the traction boundary conditions on $(-L,L)\times\partial\Omega$ in the form $(\mathbb{C}^{o}\cdot\nabla\mathbf{u}+\langle\mathbb{C}\cdot\nabla h\rangle\cdot\mathbf{w})\cdot\mathbf{n}=\mathbf{p}$ and $\langle\mathbf{R}^{2}\rangle^{-1}\cdot(\langle\mathbf{R}\cdot\mathbb{C}\cdot\nabla h\rangle\cdot\mathbf{w}+\langle h\mathbf{R}\cdot\mathbb{C}\rangle:\overline{\nabla}\,\mathbf{v})\cdot\mathbf{n}=\mathbf{q}$, where $\mathbf{v}$ is defined by (8). It can be seen that the diagonal 3×3 matrix $\mathbf{R}$ determines the periodic character of the traction oscillations on boundary $(-L,L)\times\partial\Omega$. Let the l-periodic functions $\mathbb{C}(\cdot)$, $\rho(\cdot)$ be the even functions of x_1; in this case l-periodic function $h(x_1)$ is odd, cf. Fig. 2. Under assumption that $\mathbf{R}(\cdot)$ is an even function we obtain that $\langle h\mathbf{R}\cdot\mathbb{C}\rangle=\mathbf{0}$. In this case the traction boundary conditions will take the form

$$\begin{aligned}&(\mathbb{C}^{o}\cdot\nabla\mathbf{u}+\langle\mathbb{C}\cdot\nabla h\rangle\cdot\mathbf{w})\cdot\mathbf{n}=\mathbf{p}\\&\langle \mathrm{R}^{2}\rangle^{-1}\cdot(\langle\mathrm{R}\cdot\mathbb{C}\cdot\nabla h\rangle\cdot\mathrm{w})\cdot\mathrm{n}=\mathrm{q}\end{aligned}\tag{11}$$

which has to hold in $(-L,L)\times\partial\Omega$ and for every $t\geq0$.

In order to analyze the propagation of the boundary traction fluctuations we shall restrict considerations to the thick layer bounded by planes $x_2=0$, $x_2=H$, where $H>>l$. This layer can be obtained from the prism shown in Figure 1 after setting $L\rightarrow\infty$, $H_3\rightarrow\infty$. We assume that the plane x_2=0 is subjected to the harmonic traction fluctuations given by

$$\mathbf{q}=\mathbf{q}^{o}(x_1)\cos\omega t$$

We also assume that on the boundary plane $x_2=H$ boundary traction fluctuation $\mathbf{q}$ are equal to zero. For the sake of simplicity we shall restrict ourselves to the plane-strain problem in which every function is independent on the x_3-coordinate and $u_3=w_3=0$, $q_3=0$. In this case we shall look for the solution to the second from equations (2) in the form

$$\mathbf{w}=\mathbf{w}^{o}(x_1,x_2)\cos\omega t,\quad x_2\in(0,H),\quad -\infty<x_1<\infty\tag{12}$$

It follows that function $\mathbf{w}^{o}$ has to satisfy equation

$$l^2\nabla\cdot(\langle\mathbb{C}\rangle:\overline{\nabla}\mathbf{w}^{o})-(\langle\nabla h\cdot\mathbb{C}\cdot\nabla h\rangle-\omega^2l^2\langle\rho\rangle\mathbf{1})\mathbf{w}^{o}=\mathbf{0}\tag{13}$$

where $\mathbf{1}$ is the unit tensor. Let us recall that the material components of the laminated medium are assumed to have elastic symmetry planes x_1 = const, x_2 = const. In this case we obtain

$$\begin{aligned}q_1^{o}&=-\langle(R_{11})^2\rangle^{-1}(\langle R_{11}C_{1212}\partial_1h\rangle w_2^{o})\\q_2^{o}&=-\langle(R_{22})^2\rangle^{-1}(\langle R_{22}C_{2211}\partial_1h\rangle w_1^{o})\end{aligned}$$

Subsequently we restrict considerations to the periodic piecewise constant fluctuations of the boundary tractions. In this case $\mathbf{w}=\mathbf{w}^{o}(x_2)$. Hence setting $R_{11}=$ $=R_{22}=R_{33}\partial_1h$ the boundary conditions for $\mathbf{w}^{o}$ are

$$w^o{}_1(0) = -\frac{\langle(\partial_1 h)^2\rangle}{\langle C_{2211}(\partial_1 h)^2\rangle} q^o{}_2, \qquad w^o{}_1(H) = 0$$

$$w^o{}_2(0) = -\frac{\langle(\partial_1 h)^2\rangle}{\langle C_{1212}(\partial_1 h)^2\rangle} q^o{}_1, \qquad w^o{}_2(H) = 0$$

Components $w^o{}_1, w^o{}_2$ of the vector field $\mathbf{w}^o$ will be now assumed in the form (here and hereafter $\alpha = 1, 2$; no summation over α!) $w^o{}_\alpha(x_2) = \eta_\alpha(x_2) w^o{}_\alpha(0)$, $\alpha = 1,2$ where η_α are new unknowns satisfying conditions $\eta_\alpha(0) = 1$, $\eta_\alpha(H) = 0$. Let d be a certain length parameter and define

$$\xi = \frac{x_2}{d}, \qquad \lambda = \frac{l}{d},$$

$$\gamma_\alpha = \lambda \frac{\langle C_{2\alpha 2\alpha}(\partial_1 h)^2\rangle}{\langle C_{2\alpha 2\alpha}\rangle} = 12\lambda \frac{(v'' C'_{2\alpha 2\alpha} + v' C''_{2\alpha 2\alpha})}{v' v''(v' C'_{2\alpha 2\alpha} + v'' C''_{2\alpha 2\alpha})},$$

$$(\Omega_\alpha)^2 = \frac{\omega^2 l^2 \langle\rho\rangle}{\langle C_{1\alpha 1\alpha}(\partial_1 h)^2\rangle} = \frac{\omega^2 l^2 v' v''(v'\rho' + v''\rho'')}{12(v'' C' + v' C'')}, \quad \alpha = 1,2$$

Denoting $\zeta_\alpha(\xi) \equiv \eta_\alpha(\xi d)$ we obtain from (13) two independent equations for ζ_α of the form

$$\lambda^2 \frac{d^2\zeta}{d\xi^2} - \gamma^2(1 - (\Omega_\alpha)^2)\zeta = 0 \tag{14}$$

together with the boundary conditions

$$\zeta_\alpha(0) = 1, \quad \zeta_\alpha(H/d) = 0, \quad \alpha = 1,2 \tag{15}$$

where ζ, γ, Ω stand either for ζ_1, γ_1, Ω_1 or for ζ_2, γ_2, Ω_2.

3. Numerical analysis

Now we are to the made a detailed analysis of the evolution of boundary fluctuation displacements. Firstly, we examine the character of the coefficient γ. To this end let us denote nondimensional parameters

$$\alpha \equiv C''/C', \quad v' \equiv v, \quad v'' = 1 - v, \quad 0 < v < 1 \tag{16}$$

Thus, coefficient γ depend on v i α by the following formula

$$\gamma = \gamma(v,\alpha) = 12 \frac{1 - v + v\alpha}{(1-v)v\left[v + (1-v)\right]\alpha} \tag{17}$$

Let

$$\gamma_{\min} = \inf_{\alpha\in(0,1)} \gamma(v,\alpha) = \min\left\{\frac{12}{(1-v)^2}, \frac{12}{(1-v)^2}\left(1+\frac{1-2v}{v^2}\right)\right\}$$
$$\gamma_{\max} = \sup_{\alpha\in(0,1)} \gamma(v,\alpha) = \max\left\{\frac{12}{(1-v)^2}, \frac{12}{(1-v)^2}\left(1+\frac{1-2v}{v^2}\right)\right\} \tag{18}$$

Under the above denotations it can be deduced that admissible pairs of parameters v and α have to satisfy inequalities

$$\gamma_{\min} = \frac{12}{(1-v)^2}\left(1+\frac{1-2v}{v^2}\right) \le \alpha \le \gamma_{\max} = \frac{12}{(1-v)^2}, \quad for \quad \frac{1}{2} \le v \le 1$$
$$\gamma_{\min} = \frac{12}{(1-v)^2} \le \alpha \le \gamma_{\max} = \frac{12}{(1-v)^2}\left(1+\frac{1-2v}{v^2}\right), \quad for \quad 0 \le v \le \frac{1}{2} \tag{19}$$

A few graphs of function $(0,1) \ni v \to \gamma(v) \in (0,\infty)$ for a certain α is given in Figure 3. It is clear that parameter γ_λ defined by

$$\gamma_\lambda \equiv \gamma / \lambda \tag{20}$$

takes all positive values. However, in accordance to denotations parameter γ depends exclusively on the material constants and do not depend on the microstructure length parameter l. On the other hand parameter λ is proportional to microstructure length parameter l. In the subsequent analysis the value $\gamma = 200$ is taken into account.

Let us rewrite boundary value problem under consideration in the form:

$$\frac{d^2\zeta}{d\xi^2} - \gamma_\lambda^{\,2}(1-\Omega^2)\zeta = 0$$
$$\zeta(0) = 1, \quad \zeta(H/d) = 0 \tag{21}$$

which will be fundamental for the subsequent analysis. For frequencies $\Omega\in(0,1)$ solution the above problem is give by (cf. Fig. 4)

$$\zeta = \zeta_o \exp\tilde{\sigma}(\xi_o - \xi) \tag{22}$$

where $\zeta_o = H/d$, $\tilde{\sigma} = \gamma_\lambda\sqrt{1-\Omega^2}$ and $\zeta_o \equiv 1/\sin\tilde{\sigma}\xi_o$. Here and hereafter the unit d is uniquely determined by $H/d = 1$. It can be seen that the solution $\zeta(\cdot)$ to the above boundary-value problems depends on the dimensionless frequencies Ω of boundary tractions fluctuation on $x_2 = 0$. If $0 \le \Omega < 1$ then the boundary tractions decays. If $\Omega \ge 1$ then fluctuation propagates into the layer $0 < x_2 < H$. Numerical results describing this phenomenon for $d = H$ are shown in Figure 4 and for $d = l$ in Figure 5, in both cases calculations were carried out for $\gamma = 200$. For the semi-space $H \to \infty$ and instead of condition $\zeta(H) = 0$ we have to require that the solution $\zeta(\xi)$ to equation (14) satisfies condition $\zeta(\xi) \to 0$ for $\xi \to \infty$. If $0 < \Omega < 1$ then the solution for the semispace $x_2 > 0$ exists and the boundary effect caused by oscillations of the boundary tractions rapidly decays. If $\Omega \geq 1$ then the solution for the semispace in the form governed by (14), (15) does not exist.

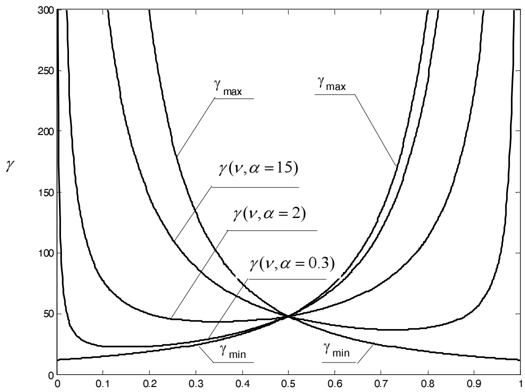

Fig. 3. Graphs of function $(0,1) \ni \nu \to \gamma(\nu) \in (0,\infty)$ for a certain parameters α

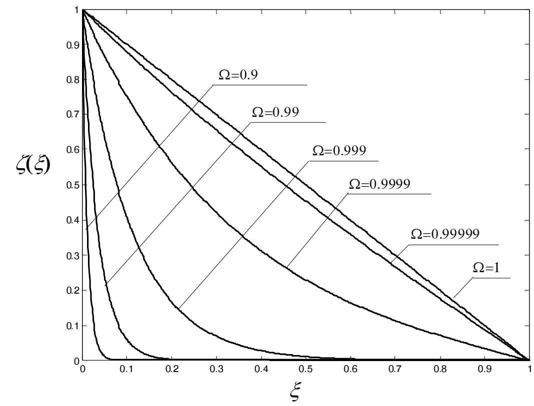

Fig. 4. Diagrams of the dimensionless fluctuation variable $\zeta(\xi)$, $\xi = x/H$, for $\Omega = (0.1)$ and $\gamma = 200$; ratio $\lambda = l/H$ is used as a parameter

It easy to verify that intrinsic fluctuations ζ monotonically decay when the distance ξ from the boundary $\xi = 0$ increasing. Moreover, if parameter $\Omega \in (0,1]$ decaying then the speed of decaying of the solution to the boundary value problem under consideration increasing. Roughly speaking, to large increments of parameter Ω in the neighbourhood of $\Omega = 0$ the fluctuation gradient related to Ω reply small increments. The quite different situation we observe in the neighbourhood of $\Omega = 1$ - to small increments of parameter Ω in the neighbourhood of $\Omega = 1$ fluctuation gradient related to Ω reply large increments.

In this case decaying of the solution is so impetuous that suitable graphs of solutions are almost indiscernible and the form of the solution is practically independent on the parameter Ω. The interrelations can be also deduced in Figure 5, where the derivative with respect to the parameter Ω of the norm $\|\zeta_1-\zeta\|$ in $C([0,H/d])$ as a function of the parameter Ω is illustrated.

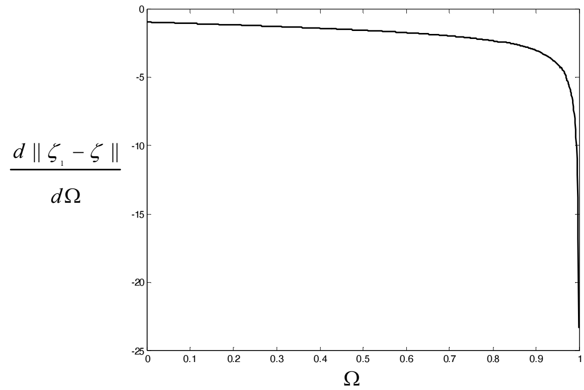

Fig. 5. The derivative with respect to the parameter Ω of the norm $\|\zeta_1-\zeta\|$ in $C([0,H/d])$ as a function of the parameter $\Omega\in(0,1)$

For nondimensional frequency $\Omega = 1$ is a part of the straight line illustrated in Figure 4 by a dot line. For $\Omega > \Omega_0 = 1$ solution to the considered problem is of the form

$$\zeta = \zeta_o \sin\sigma(\xi_o - \xi) \tag{23}$$

where

$$\zeta_o = -\frac{1}{\sin\sigma\xi_o}, \quad \sigma = \gamma_\lambda\sqrt{\Omega^2-1}, \quad \xi_o = H/d \tag{24}$$

It must be emphasized that the solution (23) exists provided that $\sin\frac{\sigma H}{d} \neq 0$. Moreover, if $\Omega > 1$ then solution (23) has at least one local ekstremal point located in the interval $0 < \xi < \xi_o$. Point $(\bar{\xi}, \zeta(\bar{\xi}))$ is an extremal point which abscissa $\bar{\xi}$ is located nearest to $\xi_o = H/d$ depends on Ω and is given by the formula

$$\bar{\xi} = \xi_o - \frac{\pi}{2\sigma} = \frac{H}{d} - \frac{\pi}{2\gamma_\lambda\sqrt{\Omega^2-1}} \tag{25}$$

Bearing in mind (24) it can be deduced that solution to the considered problem exists provided that

$$\Omega \neq \Omega_{2m}, \quad m = 1, 2, \ldots \tag{26}$$

where

$$\Omega_n \equiv \sqrt{1 + \left(\frac{n\pi}{2\gamma_\lambda \xi_o} \right)^2}, \quad n = 1, 2, \ldots \tag{27}$$

Mentioned above the sequence of nondimensional frequencies Ω_n determines the sequence of subsequent intervals $(0,\Omega_1)$, (Ω_1,Ω_2),... in which solution exists.

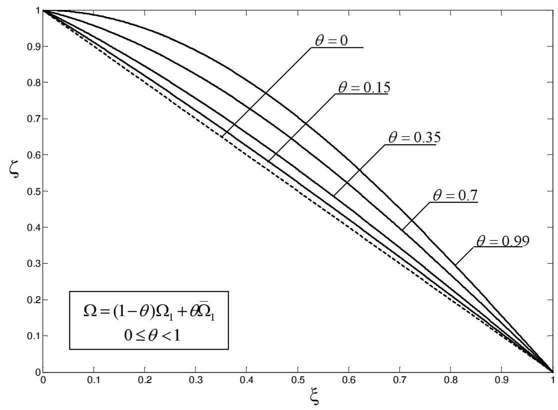

Fig. 6. Diagrams of the dimensionless fluctuation variable $\zeta(\xi)$, $\xi = x/H$ *for* $\Omega_1 < \Omega < \bar{\Omega}_1$ and $\gamma = 200$; ratio $\lambda = l/H$ is used as a parameter

It can be seen that if nondimensional frequency Ω satisfies condition $\Omega_1 \le \Omega < \Omega_2$ the solution to the boundary value problem exists. Denote

$$\bar{\Omega}_1 = \Omega_1 \equiv \sqrt{1 + \left(\frac{\pi}{2\xi_o \gamma_\lambda} \right)^2} \tag{28}$$

The above value of nondimensional frequency $\bar{\Omega}_1$ satisfies condition $\Omega_1 < \bar{\Omega}_1 < \Omega_2$. If $\Omega_1 < \Omega \le \bar{\Omega}_1$ then the boundary tractions rapidly decays, cf. Figure 6. If $\bar{\Omega}_1 \le \Omega < \Omega_1$ then the boundary tractions propagates into the layer $0 < x_2 < H$ up to the certain point $\bar{\xi}$ and then rapidly decays. For $\xi = \bar{\xi}$ $\zeta(\xi)$ has a maximal point. For $\Omega = \Omega_2$ solution does not exist. Numerical results describing this phenomenon for $d = H$ are shown in Figure 7. Straight line $\xi = \overline{\xi}_1$, where $\overline{\xi}_1 = 4\xi_o / 5$ is the first asymptotic line for the solutions $\zeta(\xi)$.

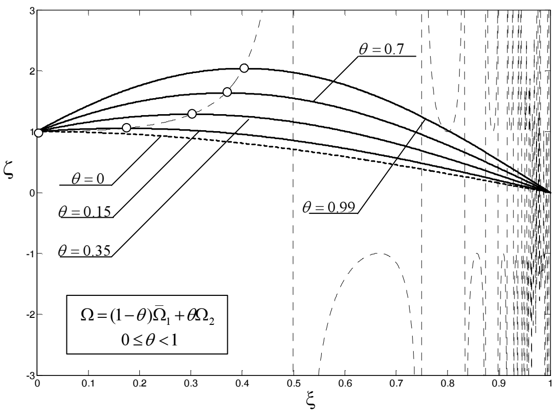

Fig. 7. Diagrams of the dimensionless fluctuation variable $\zeta(\xi)$, $\xi = x/H$ *for* $\overline{\Omega}_1 < \Omega \le \Omega_2$ and $\gamma = 200$; ratio $\lambda = l/H$ is used as a parameter

It can be seen that if nondimensional frequency Ω satisfies condition $\Omega_2 \le \Omega < \Omega_4$ the solution to the boundary value problem exists. Denote

$$\overline{\Omega}_2 = \Omega_2 \equiv \sqrt{1 + \left(\frac{3\pi}{2\xi_o \gamma_\lambda} \right)^2} \tag{29}$$

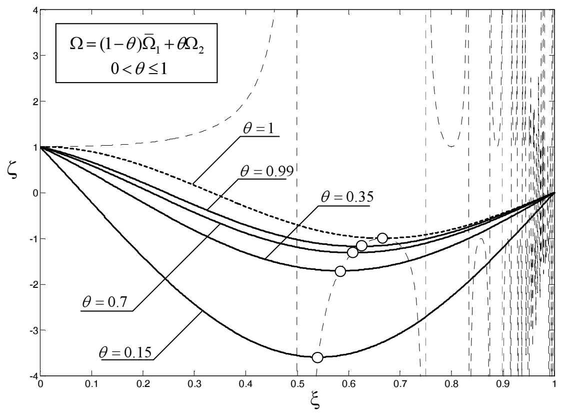

Fig. 8. Diagrams of the dimensionless fluctuation variable $\zeta(\xi)$, $\xi = x/H$ *for* $\Omega_2 < \Omega \le \overline{\Omega}_2$ and $\gamma = 200$; ratio $\lambda = l/H$ is used as a parameter

The above value of nondimensional frequency $\overline{\Omega}_2$ satisfies condition $\Omega_2 < \overline{\Omega}_2 < \Omega_4$. If $\Omega_2 < \Omega \le \overline{\Omega}_2$ then the boundary tractions rapidly decay to minimum and then increase to the boundary $\xi = 1$ where the boundary condition $\zeta(1) = 0$ is satisfied, cf. Fig. 8. If $\overline{\Omega}_2 \le \Omega < \Omega_4$ then the boundary tractions also rapidly decay to minimum and then increase to the boundary $\xi = 1$ where the boundary condition $\zeta(1) = 0$ is satisfied. However in $\Omega = \overline{\Omega}_2$ this minimum is closest to zero in the interval $\Omega_2 < \Omega < \Omega_4$. Straight line $\xi = \overline{\xi}_2$, where $\overline{\xi}_2 = 8\xi_o / 9$ is the second asymptotic line for the solutions $\zeta(\xi)$.

Presented above examples of solutions (23) to the boundary value problem (21) under consideration is a beginning fragment of classification of the set of such solutions onto the disjoint classes. The full classification will be presented in the extended version of this paper.

Concluding remarks

The new results and information on the dynamic behaviour of a laminated medium obtained in this contribution can be listed as follows:

1° The initial-boundary value problems for fluctuation variable **w**, can be formulated independently of the pertinent problem of finding averaged displacement **u.** Moreover, fluctuation variable **w** depends only on boundary and initial fluctuations of tractions and displacements, respectively.

2° If harmonic tractions are applied to the boundary of the solid which is perpendicular to the lamina interfaces, then the propagation of displacement fluctuations in a laminated medium depends on the frequency of boundary tractions. If those frequencies are sufficiently large then the displacement fluctuations propagates inside the region occupied by the medium. For small values of these frequencies we obtain a certain boundary layer effect.

3° The averaged displacement field **u** depends on the fluctuation variable gradient ∇**w** which determine the source term in equation of motion (2). Hence, if fluctuation variable **w** depends only on time then averaged displacement **u** is independent on the fluctuations of initial displacements.

References

[1] Wierzbicki E., Woźniak C., Łacińska L.. Boundary and initial fluctuation effect on dynamic behaviour of a laminated solid, Arch. App. Mech. 2005.

[2] Woźniak C., Wierzbicki E., Averaging techniques in thermomechanics of composite solids, Wyd. Politechniki Częstochowskiej, Częstochowa 2000.