USE OF FUZZY PARAMETERS FOR OPTIMAL CONTRACTS

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Abstract. Contracts, in general, can concern many aspects referring to the arrangement of rules among players. An example of such a game can be a contract between an employer and employee or between co-operating parties (coalitions) as well as a set of tender rules [21]. All these phenomena should be treated differently, since they can be different models concerning the rules and mechanisms of specific game rules. Models of contract games rely on functions or established levels of payment and frequently - on usefulness functions. For example, in the contract between the employer and the employee one uses the usefulness function in the form of [19]:

$$U(w,a) = C1 * \sqrt{w} - C2 * a$$

where: w - employee's payment, a - effort level, $C1$ and $C2$ - constants.
A well prepared contract makes the payment dependent on obtained effects and keeping to deadlines. The payment for the employer is obviously his profit obtained from employing the employee, minus the costs of employment. Payment functions can have, and as a rule they have, different shapes for employers and employees. The estimation of optimal payments, dependent on productivity, is the employer's principal problem. One certainly should take the level of acceptance into consideration [29], exceeding it makes the undertaking of the job profitable for the employee. The problem becomes more complicated by introducing fuzzy parameters or uncertain knowledge elements, but the position of the employer becomes more flexible at the same time. It results from the fact that the employer does not have to use information of experts and he can depend on the controlled parameter of optimization results that brings the solution closer to real situations or optimally brings up the level of his payment.

1. Introduction: models and parameters of contract

The game between employer and employee is partly unilateral which results from the fact that the employer and the employee have simultaneous influence exclusively on the employee's payment. However, the employer's payment is, as a rule, beyond the employee's interest (Fig. 1).
Every employee can be characterized in the accepted model by means of the diagram (Fig. 2).
Figures 3a and 3b present the patterns of employees working with high and low productivity.

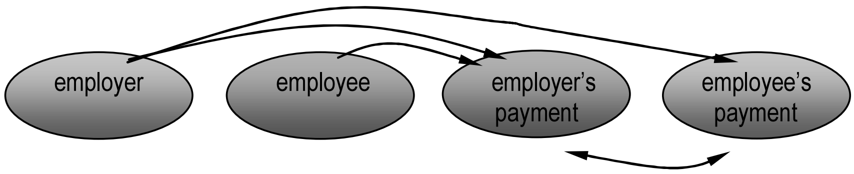

Fig. 1. Interaction and range diagram of information use in creating a contract

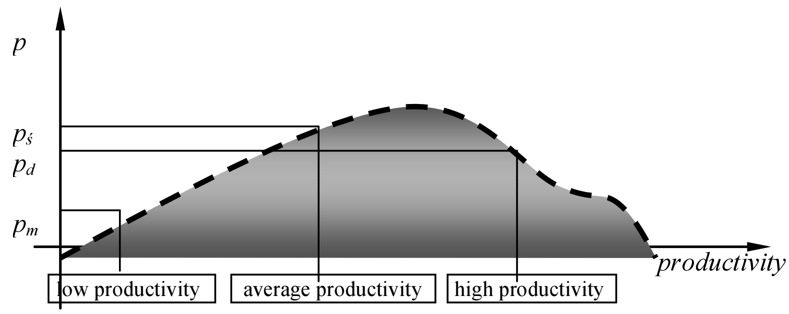

Fig. 2. Probability of doing work with different productivity levels (p)

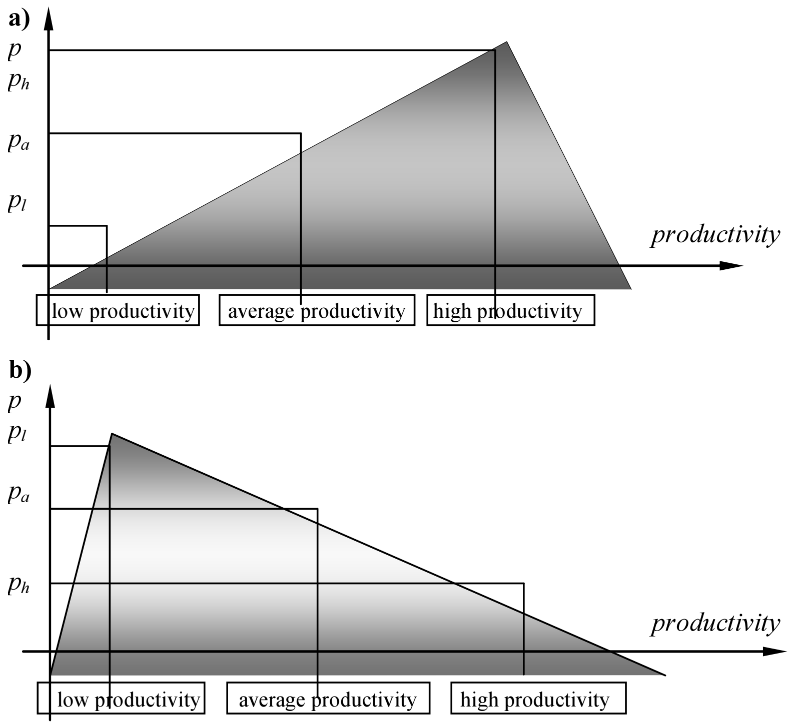

Fig. 3. Patterns of employees working with high (a) and low (b) productivity, where: p - probability of realization of tasks at three levels. Here: p_l, p_a and p_h - probabilities of work with low, average and high productivity, respectively

The expected value of the usefulness function for different and extreme levels of productivity can be estimated as follows:

1. A player working with minimal productivity is characterized by the following usefulness:

$$E(u_{min}(w,a)) = p_{min,h}*x1 + p_{min,a}*x2 + p_{min,l}*x3 - a_{min} \tag{1}$$

   where:
   $p_{min,h} = 0.05$ *(probability of work with high productivity)*
   $p_{min,a} = 0.24$ *(probability of work with average productivity)*
   $p_{min,l} = 0.71$ *(probability of work with low productivity)*
   $a_{min} = 0.5$
   Parameters are set for hypothetical patterns.

2. A player working with maximal productivity is characterized by the following usefulness:

$$E(u_{max}(w,a)) = p_{max,h}*x1 + p_{max,a}*x2 + p_{max,l}*x3 - a_{max} \tag{2}$$

   where:
   $p_{max,h} = 0.71$ *(probability of work with high productivity)*
   $p_{max,a} = 0.24$ *(probability of work with average productivity)*
   $p_{max,l} = 0.05$ *(probability of work with low productivity)*
   $a_{max} = 8$
   Parameters are set for hypothetical patterns.

Assuming the value of the usefulness function of an unemployed person, receiving an unemployment benefit in the amount of 100 currency units, as the acceptance level (threshold) *Th*, i.e.:

$$Th = U(ben,a = 0) = U(100,0) = \sqrt{w} - a = \sqrt{100} - 0 = 10 \tag{3}$$

it is possible to define the form of limitations as follows:

$$\begin{gathered} \mathrm{p}_{\mathrm{min,h}}*x1 + p_{min,a}*x2 + p_{min,l}*x3 - a_{min} \geq Th \\ p_{max,h}*x1 + p_{max,a}*x2 + p_{max,l}*x3 - a_{max} \geq p_{min,h}*x1 + p_{min,a}*x2 + p_{min,l}*x3 - a_{min} \end{gathered} \tag{4}$$

The criterion, i.e. goal function, can be defined from the employer's point of view, since he will draw up the contract in this example. So, he will want to employ an employee working with high productivity and, being guided by sheer greed and short-sightedness, he will want to pay the employee as little as possible. In this connection, he will use the following structure of expression:

$$E(w) = p_{max,h}*x1^2 + p_{max,a}*x2^2 + p_{max,l}*x3^2 \rightarrow min \tag{5}$$

In this example (data from (1), (2), (3)), using Solver's optimization [13] or classical methodology, consisting in the calculation of partial derivatives from limitations and the goal function [27], one obtains the following results:

Table 1

Results of quadratic optimization by estimation of payment components for employer

p1	x1 low	p2	x2 average	p3	x3 high	a	acceptation level	E(U)	
0.71	7.383971	0.24	18	0.05	18.74761	0.5	10	10	min. productivity
0.05	7.383971	0.24	18	0.71	18.74761	8	10	10	max. productivity
330.0318	component 1 of payment		component 2 of payment		component 3 of payment				
E(w)>(5)	54.52303		324		351.4728				

Interpretation of the results is the following: if the employee works with minimal productivity then he receives 54.52 c.u. (currency units), if with average productivity - 54.52 + 324 = 378 c.u. and if with high productivity - 54.52 + 351.47 = = 405.99 c.u. Thus, one takes the first component as the constant and the two remaining components as dynamic quantities dependent on the productivity. Certainly, it is easy to imagine that there are far more productivity thresholds, but the consideration of possible cases is not the aim of this publication.

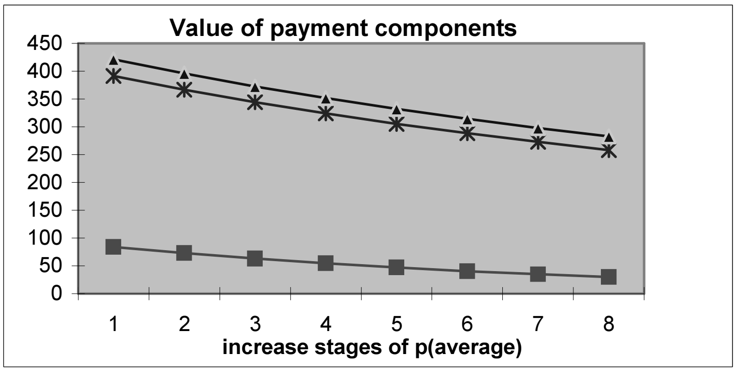

Fig. 4. Influence of requirements (Table 2), for average productivity, on amount of payment components

However, the selection of the $p_{min,h}$, $p_{min,a}$, p_{minl}, $p_{max,h}$, $p_{max,a}$, $p_{max,l}$ probabilities can be an interesting issue. The change in values of these probabilities can be a subject of payment manipulations either for the employer's or employee's advantage. Since, it is necessary to find a measurable justification of the increase in probability p_a, for example. A depreciation of all the components of payments appears, as shown by the successive optimizations for increasing levels of p_a = {0.15; 0.18; 0.21, 0.24; 0.27; 0.30; 0.33; 0.36} (Fig. 4). One should interpret it in this way that the increase in expected, standard requirements, concerning average productivity, ensues with the increase in p_a that influences the increase in the employee's payments.

Table 2

Relationships between payment components and $p_a = p$(average)

component 1 of payment		component 2 of payment		component 3 of payment	p(average)
83.98239		391.2571		421.3917	0.15
72.81051		366.6818		395.8724	0.18
63.05428		344.3512		372.6563	0.21
54.52303		324		351.4728	0.24
47.05547		305.4011		332.09	0.27
40.51454		288.3589		314.3082	0.3
34.78325		272.7043		297.9549	0.33
29.76138		258.2908		282.88	0.36

Similar correction of the remaining probabilities (p_l, p_h) causes analogical (i.e. digitized) modifications of the usefulness function.
The consideration of linear models leads to comparable conclusions.

2. Fuzziness of parameters of contract models and aspects of their optimal selection

The contract model will be presented in a possible general form to introduce fuzzy elements. One can also introduce simplifications consisting in, for example, the removal of limitations resulting from probabilistic characteristics. It is justified by fuzziness strategies in which one allows the possibility of crossing the sum level unit of usefulness function values for the productivity components [24]. Analogies between the probabilistic, fuzzy and interval strategies are presented in publications [14]. The general form of the usefulness and limitation functions can be as follows:

$$[E(w)] = [\mu 1_{max,k}]*x1^{[\beta]} + [\mu 2_{max,k-1}]*x2^{[\beta]} + [\mu 3_{max,k-2}]*x3^{[\beta]} + \ldots \rightarrow \min$$

$$\begin{cases} [\mu 1_{min,k}]*x1^{[\beta-1]} + [\mu 2_{min,k-1}]*x2^{[\beta-1]} + [\mu 3_{min,k-2}]*x3^{[\beta-1]} + \ldots - [a_{min}] \geq [Th] \\ [\mu 1_{1,k}]*x1^{[\beta-1]} + [\mu 2_{1,k-1}]*x2^{[\beta-1]} + [\mu 3_{1,k-2}]*x3^{[\beta-1]} + \ldots - [a_1] \geq \\ \quad \geq [\mu 1_{min,k}]*x1^{[\beta-1]} + [\mu 1_{min,k-1}]*x2^{[\beta-1]} + [\mu 1_{min,k-2}]*x3^{[\beta-1]} + \ldots - [a_{min}] \\ [\mu 1_{2,k}]*x1^{[\beta-1]} + [\mu 2_{2,k-1}]*x2^{[\beta-1]} + [\mu 3_{2,k-2}]*x3^{[\beta-1]} + \ldots - [a_2] \geq \\ \quad \geq [\mu 1_{1,k}]*x1^{[\beta-1]} + [\mu 1_{1,k-1}]*x2^{[\beta-1]} + [\mu 1_{1,k-2}]*x3^{[\beta-1]} + \ldots - [a_1] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [\mu 1_{max,k}]*x1^{[\beta-1]} + [\mu 2_{max,k-1}]*x2^{[\beta-1]} + [\mu 3_{max,k-2}]*x3^{[\beta-1]} + \ldots - [a_{max}] \geq \\ \quad \geq [\mu 1_{max-1,k}]*x1^{[\beta-1]} + [\mu 1_{max-1,k-1}]*x2^{[\beta-1]} + [\mu 1_{max-1,k-2}]*x3^{[\beta-1]} + \ldots - [a_{max-1}] \end{cases} \quad (6)$$

In the above model fuzziness concerns the four groups of parameters: [$\mu t_{i,j}$], [a_i], [Th], [β]. The result of the optimization procedure can be a set of heuristics allowing a simple comparison of the most effective values of the parameters for both players. The basis for creating these heuristics will be the analysis results of optimal relationships between payment component values and fuzzy parameter changes. One of such analyses concerns the influence of the acceptation level on the employee's payment (Fig. 5).

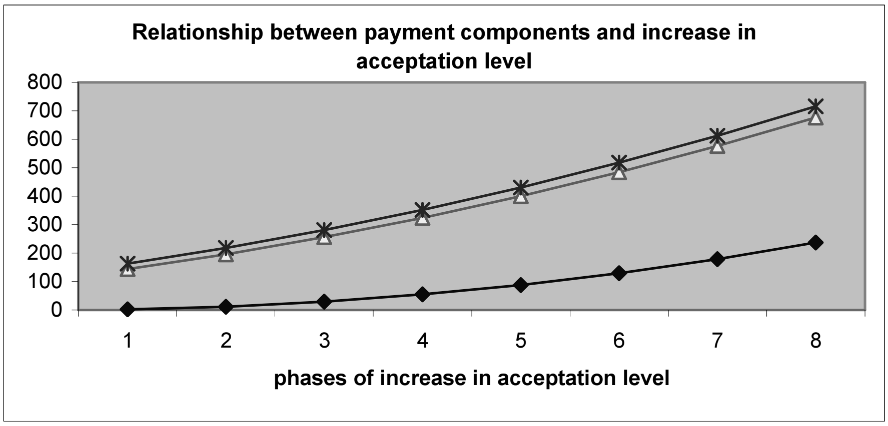

Fig. 5. Relationship between payment component values for employee and increase in acceptation level (threshold) (Th = 7, 8, 9, 10, 11, 12, 13, 14)

The next investigation of the influence on the employee's payment is the analysis of influence requirements concerning efforts (minimal, maximal and intermediate levels) $a_{min}, a_1, a_2, \ldots, a_{max}$ on payment component values (Figures 6 and 7).

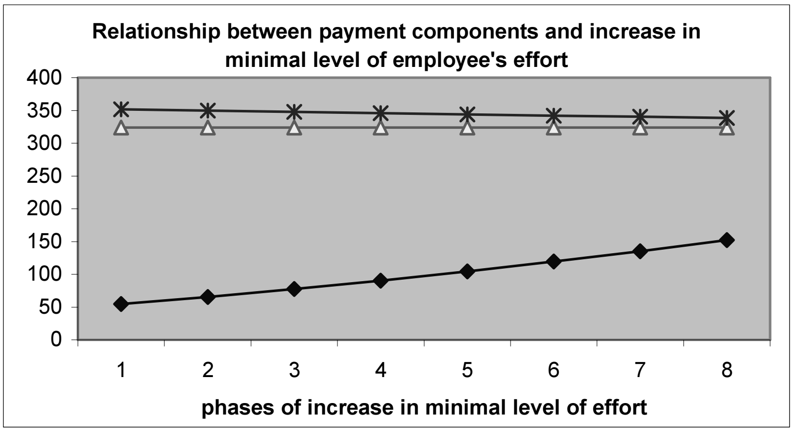

Fig. 6. Relationship between payment component values for employee and set level of minimal effort a_{min} (a_{min} = 0.5; 1.0; 1.5; 2.0; 2.5; 3.0; 3.5; 4.0)

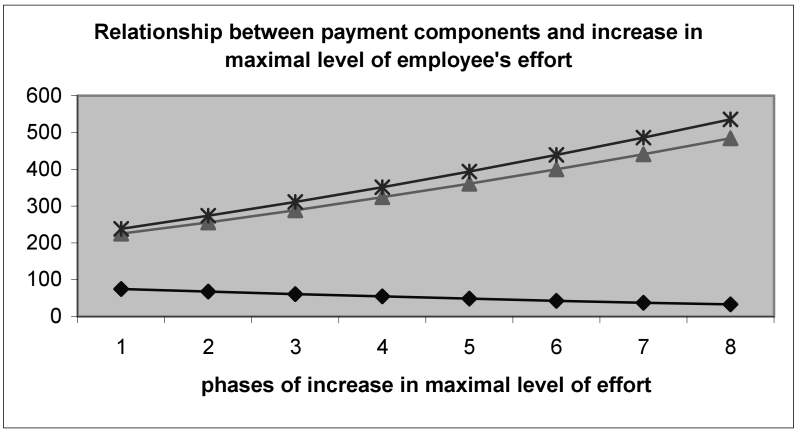

Fig. 7. Relationship between payment component values for employee and set level of maximal effort a_{max} (a_{max} = 5, 6, 7, 8, 9, 10, 11, 12)

All that is left to do is to investigate the influence of polynomial degree of the criterion function on the payment components (Fig. 8).

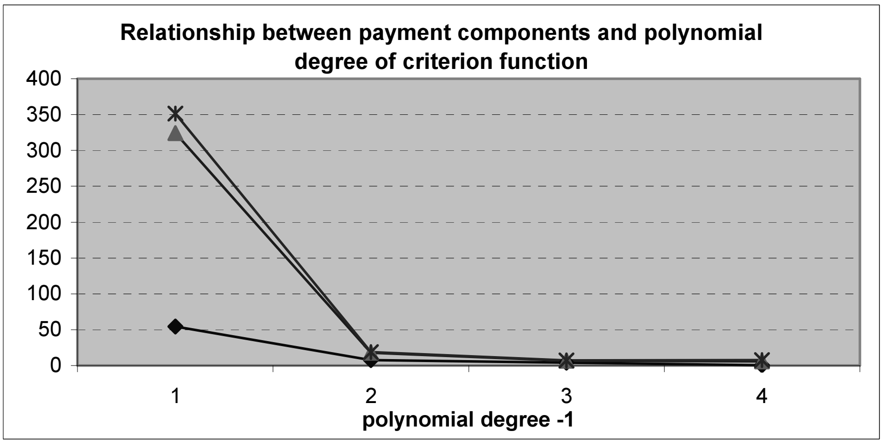

Fig. 8. Relationship between payment component values for employee and polynomial degree of criterion function β = 2, 3, 4, 5

The influence of model complexity (degree of approximating polynomial) on the payments for the employee as well as for the employer is unfavourable. In the case of the employer, this results from the fact that the increase in the model complexity causes a decline in stimulation values of the contract structure. It is the consequence of a decrease in the relation of the payment components making a bonus for average and high productivity with reference to the base payment component: (component 2 of payment/component 1)↓, (component 3 of payment/component 1)↓.

The creation of heuristics is realized on the plane of diagrams of relationships between optimal components and parameters (Fig. 9).

a) b)

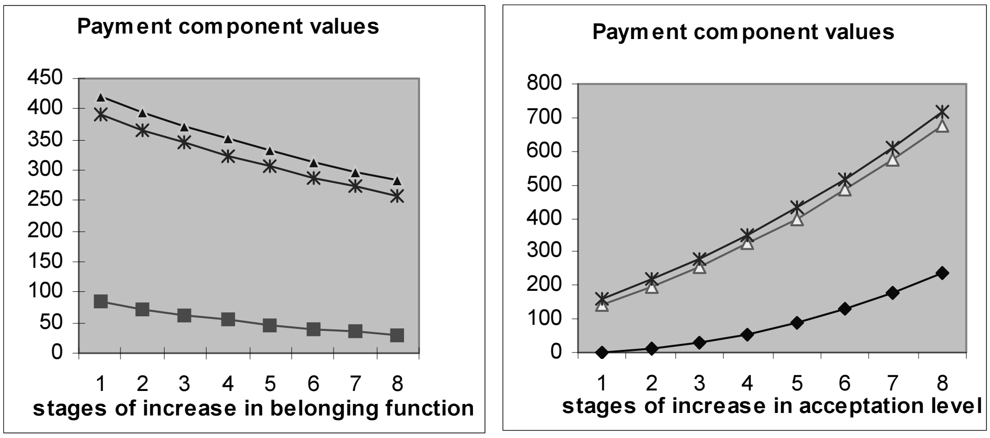

Fig. 9. Placement of fuzzy parameters in diagrams of payment components

The creation strategies of the heuristics can be described as follows:

a) selection of player,
b) definition of favourable tendency of influence of parameter change on contract efficiency (i.e. on values of payment components),
c) selection of fuzzy interval limits closest to optimal solution,
d) arrangement of conditions (circumstances and conclusions).

For example, selecting the employee as the player and assuming fuzziness of the two parameters from Figure 9 ([μ t] = [0.20; 0.23]; [Th] = [6.0; 6.4]), one creates the following heuristics:

select $opt\_\mu t = \left\{ \mu t \in \left[\underline{\mu}, \overline{\mu}\right] \middle| \max\left(p.c\left(\underline{\mu}\right), p.c\left(\overline{\mu}\right)\right) \right\}$ | (Fig. 9a),

$opt\_\mu\, t = \underline{\mu} = 0.20$

select $opt\_\ a = \left\{ a \in \left[\underline{a}, \overline{a}\right] \middle| \max\left(p.c\left(\underline{a}\right), p.c\left(\overline{a}\right)\right) \right\}$ (Fig. 9b),

$opt\_a = \overline{a} = 0.64$,

where p.c denotes payment component.

Conclusions

1. The increase in the payment component levels for the employee leads, as a rule, to the decrease of the payments for the employer, however, it does not always have to be like that, since it can also involve the increase in stimulation proper-

ties for higher productivity. It takes place when the increase pace of the second payment component (taking the achievement of medium productivity level into consideration) is greater than the increase pace of the first payment component, and when the increase pace of the third payment component (taking the achievement of high payment level into consideration) is greater than the increase pace of the second component.
2. The selection of optimal limits of fuzziness of controlled parameters can be based on the retrieval of the nearest, the most favourable for the given player, elements of fuzzy sets whose location was established on the basis of change trends of the payment components for the employee (Fig. 9).
3. The increase of the contract model complexity by the increase of the polynomial degree of the criterion function (5), (6) leads to the decrease of virtues stimulating the contract efficiency (Fig. 8).
4. As was expected, the increase in the acceptation level [3] leads to the increase of the employee's payments but, at the same time, it leads to the virtues stimulating the contract efficiency (Fig. 5), which is favourable for the employer. If this profit is greater than the increase of the employee's payment then the employer's payment increases as well.

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