ON THE ISOTROPIC PROPERTIES OF A HEXAGONAL - TYPE RIGID CONDUCTOR

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Abstract. A new macroscopic model for the non-stationary heat transfer processes in a periodic hexagonal-type anisotropic rigid conductor is formulated. The main aim of this contribution is to show that the macroscopic properties of this conductor are transversally isotropic. The tolerance averaging technique as a tool of macroscopic modelling is taken into account [1].

Introduction

In this note a new tolerance averaged model for the non-stationary heat transfer processes in hexagonal-type rigid conductor is proposed. The main result of this note is to show that, under a certain condition, the overall properties of the conductor under consideration are transversally isotropic even if constituents of a conductor are anisotropic. Moreover, we are to show that in a certain special case of the introduced transversally isotropic model of the conductor under consideration a certain averaged scalar parameter called averaged relaxation time can be defined. This parameter plays a similar role as a classical relaxation time for the hyperbolic heat transfer problems in the case of homogeneous conductors. The general form of the averaged equations will be specified in order to describe the hexagonal-type periodic rigid conductor which material and geometrical properties are invariant over under $2\pi/3$-rotation with respect to the center of an arbitrary hexagonal cell. The scope of this paper is restricted to the formulation of $3D$-nostationary hyperbolic heat propagation model equations.

A certain transversally isotropic model for heat propagation in the hexagonal-type rigid conductors was formulated in [2]. However, in this paper the Fourier heat propagation law is taken into account. Results obtained in this note will be a certain generalization of those obtained in [2] to the case of hyperbolic heat transfer in aforementioned conductors based on the Cattaneo heat propagation law. Similar problems for periodic lattice-type conductors was discussed in [3] and in the framework of hexagonal-type conductors in [4].

1. Denotations

Throughout the paper we use the superscripts A,B which run over 1,...,N, superscripts a,b which run over 1,...,n and subscripts r,s which run over 1, 2, 3, summation convention with respect to these indices is assumed to hold. Let Oz be an axis normal to the periodicity Ox_1x_2 -plane. In the direction of the z-axis the conductor will be treated as homogeneous. At the same time every plane z = const. is assumed to be a plane of a material symmetry. Symbol ∂ stands for the partial derivative with respect to the z-coordinate and $\overline{\nabla}$ is a gradient with respect to $\mathbf{x} = (x_1, x_2)$. An arbitrary hexagonal cell of the composite under consideration will be denoted by $\Delta, \Delta \subset R^2$. By $\langle f \rangle(\mathbf{x})$ we denote the integral mean value over $\Delta(\mathbf{x}) = \mathbf{x} + \Delta,\ \mathbf{x} \in R^2$ of an arbitrary integrable function $f(\cdot)$ defined in R^2.

2. Formulation of the problem

The starting point of the considerations is the well known 3D-heat transfer equation in the honeycomb type periodic conductor which in this note will be rewritten in the form

$$\overline{\nabla}\overline{\mathbf{q}}(\mathbf{x},z,t) + \partial q_3(\mathbf{x},z,t) - c(\mathbf{x},z)\dot{\theta}(\mathbf{x},z,t) = f(\mathbf{x},z,t) \tag{1}$$

where $\mathbf{q} = (\overline{\mathbf{q}}, q_3)$ is the total flow and $\theta(\cdot,z,t),\ z \in R,\ t \in (t^0, t^1)$, is the temperature field and $f(\cdot,z,t)$ represents a heat sources. In order to describe the hyperbolic heat propagation in the periodic conductor under consideration we shall apply a following Cattaneo-type constitutive law

$$\mathbf{q}(\mathbf{x},z,t) + \tau\dot{\mathbf{q}}(\mathbf{x},z,t) = -\mathbf{A}(\mathbf{x})\overline{\nabla}\theta(\mathbf{x},z,t) \tag{2}$$

where τ is a relaxation time and $\mathbf{A}$ is a positive definite symmetric matrix representing an anisotropic conductivity tensor assumed in the form

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} \overline{\mathbf{A}}(\mathbf{x}) & 0 \\ 0 & A_{33} \end{bmatrix} \tag{3}$$

for A_{33} = const and $\mathbf{x} \in R^2$. Moreover, according to the periodicity of the hexagonal structure under consideration, we assume that $\mathbf{A}(\cdot)$ and $c(\cdot)$, are Δ-periodic functions and that they for composite materials attain constant values in every constituent of the conductor.

The aim of the note is to formulate a certain averaged model of the heat propagation described in on a micro-level by equation (1) and Cattaneo-type constitutive law given by (2).

3. Tool of modelling

As a tool of modelling the tolerance averaging technique will be taken into account. To this end we shall assume that the temperature field has the form

$$\theta(\mathbf{y},z,t)=\theta^0(\mathbf{y},z,t)+g^A(\mathbf{y})W^A(\mathbf{x},z,t),\quad \mathbf{y}\in\Delta(\mathbf{x}) \tag{4}$$

where averaged temperature field $\theta^0(\cdot,z,t)=\langle c\rangle^{-1}\langle c\theta\rangle(\cdot,z,t)$ and fluctuation variables $W^A(\cdot,z,t)$ are assumed to be slowly varying functions for every $(z,t)\in R\times(t^0,t^1)$ and represent a new basic unknowns. Moreover, $g^A(\cdot),\ A=1,...,N,$ are postulated *a priori* Δ-periodic functions, usually called *shape functions*, and have to satisfy conditions $\langle g^A\rangle=0, \langle cg^A\rangle=0$ and $\langle \mathbf{A}g^A\rangle=0.$ We also define the averaging operation

$$\langle f\rangle(\mathbf{x})=\frac{1}{|\Delta|}\int_{\Delta(\mathbf{x})} f(\mathbf{y})d\mathbf{y} \tag{5}$$

where f is an arbitrary integrable function. It can be proved, cf. [2], that under the above denotations the following system of equations with constant coefficients

$$\begin{aligned}
&\overline{\nabla}\cdot\langle\overline{\mathbf{A}}\rangle\cdot\overline{\nabla}\theta^0+\langle A_{33}\rangle\partial^2\theta-\langle c\rangle\dot{\theta}^0-\langle\tau c\rangle\ddot{\theta}^0+\overline{\nabla}\cdot\langle\overline{\mathbf{A}}\cdot\overline{\nabla}g^A\rangle W^A=\langle f+\tau\dot{f}\rangle\\
&\langle cg^Ag^B\rangle\dot{W}^B+\langle\tau cg^Ag^B\rangle\ddot{W}^B+\langle\overline{\nabla}g^A\cdot\overline{\mathbf{A}}\cdot\overline{\nabla}g^B\rangle W^B-\langle A_{33}g^Ag^B\rangle\partial^2W^B+\\
&\qquad+\langle\overline{\nabla}g^A\cdot\overline{\mathbf{A}}\rangle\cdot\overline{\nabla}\theta^0=-\langle(f+\tau\dot{f})g^A\rangle
\end{aligned} \tag{6}$$

holds. Since the shape functions depend on the period length l and satisfy conditions $g^A(\cdot)\in O(l)$, $l\nabla g^A(\cdot)\in O(l)$, where $O(l)$ is the known ordering symbol, coefficients $\langle\mathbf{A}g^Ag^B\rangle$ and $\langle cg^Ag^B\rangle$ in Eqs. (4) are of an order l^2 and hence the above model equations describe the effect of microstructure size on the averaged properties of the conductor. It is worthy noting that the known homogenized models of a periodic solid is not able to describe the above effect. In the next section we are to formulate a certain necessary conditions under which model equations (6) are isotropic.

4. Basic assumption

Let us denote by G the system of all shape functions taking into account in every special problem. Moreover, let $\mathbf{Q}$ represents a pertinent $2\pi/3$-rotation related to the certain center of the representative periodicity cell and let

$G_{\mathbf{Q}} \equiv \{g_{\mathbf{Q}}(\cdot): g(\cdot) \in G\}$ for $g_{\mathbf{Q}}(\mathbf{y}) \equiv g(\mathbf{Qy})$, $\mathbf{y} \in \Delta$. In this contribution we shall restrict our considerations to the case of hexagonal-type composites satisfying the following two assumptions.

***Assumption* 1**. *The material structure of the conductor is invariant under $2\pi/3$-rotations with respect to the center of an arbitrary hexagonal periodicity cell; it is mean that* $\bar{\mathbf{A}}(\mathbf{Qy}) = \mathbf{Q}\bar{\mathbf{A}}(\mathbf{y})\mathbf{Q}^T$, $c(\mathbf{Qy}) = c(\mathbf{y})$ *for every* $\mathbf{y} \in \Delta$.

***Assumption* 2**. *The set G is invariant under $2\pi/3$-rotations with respect to the center of an arbitrary hexagonal periodicity cell; it is mean that* $G_{\mathbf{Q}} \subset G$.

In the next section we are to discuss some consequences of the above assumptions.

5. Analysis of the isotropic properties of the model equations

Now we shall outline the line of approach leading from Eqs. (6) to the transversally isotropic averaged model equations. To this end we shall introduce a new enumeration of the shape functions and fluctuation variables. Let us observe that the natural consequence of the *Assumption* 2. is that the system *G* of all shape functions is divided onto the disjoint sum $G = G_1 \cup G_2 \cup ... \cup G_n$ of classes G_a, $a = 1,...,n$. Every class G_a consists either by one or three elements. Hence, in every class G_a, shape functions will be denoted by g_1^a, g_2^a, g_3^a and interconnected by formulas $g_2^a = (g_1^a)_{\mathbf{Q}}$, $g_3^a = (g_2^a)_{\mathbf{Q}}$. At the same time let W_1^a, W_2^a, W_3^a denote fluctuation variables related to shape functions g_1^a, g_2^a, g_3^a, respectively. If G_a consists of only one element we have $g_1^a = g_2^a = g_3^a$ and we admits situations, in which one shape function posses three different indices. At the same time formula (4), for every $z \in R,\, t \in (t^0, t^1)$, will be rewritten in the form

$$\begin{aligned} \theta(\mathbf{x},z,t) = \theta^0(\mathbf{x},z,t) + g_1^a(\mathbf{y})W_1^a(\mathbf{x},z,t) + g_2^a(\mathbf{y})W_2^a(\mathbf{x},z,t) + \\ + g_3^a(\mathbf{y})W_3^a(\mathbf{x},z,t), \quad \mathbf{y} \in \Delta(\mathbf{x}) \end{aligned} \tag{7}$$

Bearing in mind a representation (6) of temperature field we are to introduce new fluctuation variables. To this end define $\mathbf{t}^1 = [1,0]$, $\mathbf{t}^2 = \left[-1/2, \sqrt{3}/2\right]$, $\mathbf{t}^3 = \left[-1/2, -\sqrt{3}/2\right]$, and $\tilde{\mathbf{t}}^1 = \in \mathbf{t}^1$, $\tilde{\mathbf{t}}^2 = \in \mathbf{t}^2$, $\tilde{\mathbf{t}}^3 = \in \mathbf{t}^3$, where $\in$ denotes the Ricci tensor, i.e. tensor related to the $\pi/2$-rotation. Now we can introduce new variables

$$U^a = W_1^a + W_2^a + W_3^a,\ \mathbf{V}^a = \mathbf{t}^1 V_1^a + \mathbf{t}^2 V_2^a + \mathbf{t}^3 V_3^a,\ a = 1,...,n \tag{8}$$

It can be proved that (8) is an invertible transformation and using the formula

$$W_1^a = \mathbf{t}^1\mathbf{V}^a + U^a/3,\ W_2^a = \mathbf{t}^2\mathbf{V}^a + U^a/3,\ W_3^a = \mathbf{t}^3\mathbf{V}^a + U^a/3,\ a = 1,...,n \tag{9}$$

we can determine new fluctuation variable fields $U^a(\cdot), \mathbf{V}^a(\cdot)$ by old fluctuation variable fields $W_1^a(\cdot), W_2^a(\cdot), W_3^a(\cdot)$. Introducing variables (8) into the model equations (6), we obtain the following alternative form of model equations:

$$
\begin{aligned}
&\bar{\nabla}\cdot\langle\mathbf{A}\rangle\cdot\bar{\nabla}\theta^0+\left[\mathbf{B}^a\right]:\bar{\nabla}\mathbf{V}^a+\langle A_{33}\rangle\partial^2\theta^0-\langle c\rangle\dot{\theta}^0-\langle\tau c\rangle\ddot{\theta}^0=\langle f+\tau\dot{f}\rangle\\
&\bar{h}_2^{ab}\ddot{U}^b+\bar{c}_2^{ab}\dot{U}^b+\bar{a}_2^{ab}U^b+\bar{a}_3^{ab}\partial^2U^b=\left[f^a\right]\\
&\mathbf{H}_2^{ab}\ddot{\mathbf{V}}^b+\mathbf{C}_2^{ab}\dot{\mathbf{V}}^b+\mathbf{A}_2^{ab}\mathbf{V}^b+\left[\mathbf{B}^a\right]^T\bar{\nabla}\theta^0+\mathbf{A}_3^{ab}\partial^2\mathbf{V}^b=-\mathbf{f}^a
\end{aligned}
\tag{10}
$$

The term $\left[\mathbf{B}^a\right]:\bar{\nabla}\mathbf{V}^a$ in the first from Eqs. (10) will be called the fluctuation term. By applying a similar approach to that introduced in [3] it can be proved that coefficients $\mathbf{A}_2^{ab}$, $\mathbf{A}_3^{ab}$, $\mathbf{C}_2^{ab}$ and $\mathbf{H}_2^{ab}$ are isotropic and have the following isotropic representations:

$$
\begin{aligned}
&\mathbf{A}_2^{ab}=\hat{a}_2^{ab}\mathbf{1}+\tilde{a}_2^{ab}\in^T, \quad \mathbf{A}_3^{ab}=\hat{a}_3^{ab}\mathbf{1}+\tilde{a}_3^{ab}\in^T,\\
&\mathbf{C}_2^{ab}=\hat{c}_2^{ab}\mathbf{1}+\tilde{c}_2^{ab}\in^T, \quad \mathbf{H}_2^{ab}=\hat{h}_2^{ab}\mathbf{1}+\tilde{h}_2^{ab}\in^T,
\end{aligned}
\tag{11}
$$

for a certain pairs $\left(\hat{a}_2^{ab},\tilde{a}_2^{ab}\right),\left(\hat{a}_3^{ab},\tilde{a}_3^{ab}\right),\left(\hat{c}_2^{ab},\tilde{c}_2^{ab}\right),\left(\hat{h}_2^{ab},\tilde{h}_2^{ab}\right)$ of constants. At the same time the mean value $\langle\mathbf{A}\rangle$ of conductivity tensor $\mathbf{A}$ is transversally isotropic and has the following isotropic representation

$$
\langle\bar{\mathbf{A}}\rangle=a\mathbf{1}
\tag{12}
$$

for a certain positive constant α. Moreover the fluctuation term has the following isotropic representation

$$
\left[\mathbf{B}^a\right]:\bar{\nabla}\mathbf{V}^a=\left(\hat{b}^a1+\tilde{b}^a\in\right):\bar{\nabla}\mathbf{V}^a
\tag{13}
$$

for a certain pair $\left(\hat{b}^a,\tilde{b}^a\right)$ of constants. Now, bearing in mind results (11), (12), (13) we arrive at the equivalent form of model equations (10)

$$
\begin{aligned}
&a\bar{\nabla}^2\theta^0+\hat{b}^a\bar{\nabla}\cdot\mathbf{V}^a+\tilde{b}^a\bar{\nabla}\cdot\left(\in\mathbf{V}^a\right)+\langle A_{33}\rangle\partial^2\theta^0-\langle c\rangle\dot{\theta}^0-\langle\tau c\rangle\ddot{\theta}^0=\langle f\rangle\\
&\bar{h}_2^{ab}\ddot{U}^b+\bar{c}_2^{ab}\dot{U}^b+\bar{a}_2^{ab}U^b+\bar{a}_3^{ab}U^b_{,33}=f^a\\
&\left(\hat{h}_2^{ab}\mathbf{1}+\tilde{h}_2^{ab}\in^T\right)\ddot{\mathbf{V}}^b+\left(\hat{c}_2^{ab}\mathbf{1}+\tilde{c}_2^{ab}\in^T\right)\dot{\mathbf{V}}^b+\left(\hat{a}_2^{ab}\mathbf{1}+\tilde{a}_2^{ab}\in^T\right)\mathbf{V}^b+\\
&\qquad+\left(\hat{a}_3^{ab}\mathbf{1}+\tilde{a}_3^{ab}\in^T\right)\partial^2\mathbf{V}^b+\left(\hat{b}_2^{ab}\mathbf{1}+\tilde{b}_2^{ab}\in^T\right)\bar{\nabla}\theta^0=-\mathbf{f}^a
\end{aligned}
\tag{14}
$$

for a certain fields f^α and $\mathbf{f}^\alpha$.

6. The concept of an averaged relaxation time

Now, we shall pass to a certain special case of the presented model (14). Namely we shall restrict considerations to the case in which:

(1) heat propagates exclusively in the periodicity plane, i.e. $\theta^0 = \theta^0(\mathbf{x}, t)$, $U^a = U^a(\mathbf{x}, t)$, $\mathbf{V}^a = \mathbf{V}^a(\mathbf{x}, t)$, $a = 1, ..., n$,

(2) heat sources *f* are slowly varying functions, i.e. $f^a = 0$ and $\mathbf{f}^a = 0$,

(3) the periodicity cell has a threefold symmetry axis, i.e. $\tilde{b}^a = 0, \tilde{h}_2^{ab} = 0, \tilde{c}_2^{ab} = 0$ and $\tilde{a}_2^{ab} = 0$.

In a just mentioned special case model equations (14) takes the simpler form

$$
\begin{aligned}
& a\overline{\nabla}^2\theta^0 + \left[\hat{b}^a\right]\overline{\nabla}\cdot\mathbf{V}^a + \left[\tilde{b}^a\right]\overline{\nabla}\cdot\left(\in\mathbf{V}^a\right) - \langle c\rangle\dot{\theta}^0 - \langle\tau c\rangle\ddot{\theta}^0 = \langle f\rangle \\
& \overline{h}_2^{ab}\ddot{U}^b + \overline{c}_2^{ab}\dot{U}^b + \overline{a}_2^{ab}U^b = 0 \\
& \hat{h}_2^{ab}\ddot{\mathbf{V}}^b + \hat{c}_2^{ab}\dot{\mathbf{V}}^b + \hat{a}_2^{ab}\mathbf{V}^b + \hat{b}^a\overline{\nabla}\theta^0 = \mathbf{0}
\end{aligned}
\tag{15}
$$

Applying the formal limit passage $l \to 0$, we arrive at algebraic equations for fluctuation variables $\mathbf{V}^a$

$$
\hat{a}_2^{ab}\mathbf{V}^b + \hat{b}^a\overline{\nabla}\theta^0 = \mathbf{0} \tag{16}
$$

It can be shown that this system has a unique solution for $\mathbf{V}^a$ and first and second model equations reduce to a single heat propagation equation

$$
a^{eff}\overline{\nabla}^2\theta^0 - \langle c\rangle\dot{\theta}^0 - \langle\tau c\rangle\ddot{\theta}^0 = \langle f\rangle \tag{17}
$$

in which $a^{eff} = a - \hat{b}^a\hat{b}^b M^{ab}$ and M^{ab} is defined by $M^{ab}\hat{a}_2^{ac} = \delta^{ac}$. In the case under consideration we can introduce constant

$$
\tau_0 = \langle\tau c\rangle / \langle c\rangle \tag{18}
$$

which will be called *the averaged relaxation time* of the hexagonal-type conductor under consideration. Let

$$
\vartheta = \theta^0 e^{-\frac{t}{2\tau_0}} \tag{19}
$$

be the modified temperature field. Hence, Eqs. (17) can be written in the form of the following hyperbolic-type heat propagation equation

$$
a^{eff}\overline{\nabla}^2\vartheta^0 - \langle c\rangle\left(\frac{1}{4\tau_0}\vartheta^0 - \tau_0\ddot{\vartheta}^0\right) = \left(\langle f\rangle + \tau_0\langle f\rangle\right)e^{\frac{t}{2\tau_0}} \tag{20}
$$

It can be emphasized that if all constituents have the same relaxation time, i.e. $\tau = \tau(\mathbf{x}) = \text{const.}$, then $\tau_o = \tau$.

Conclusions

The model equations (14) represent an alternative form of the averaged model (6) of the hyperbolic heat transfer in hexagonal-type rigid conductors. Model equations (14) are transversally isotropic and hence we have proved the main thesis of this note that the averaged heat transfer response of the hexagonal-type rigid conductors is transversally isotropic. We emphasize that the in a general case components of the periodic conductor under consideration can be anisotropic in every symmetry plane $z = \text{const.}$

The obtained equations (14) include as a special case hexagonal-type conductors for which every cell has a threefold axis of symmetry in every plane $z = \text{const.}$ In this case in Eqs. (14) terms involving the Ricci tensor drop out. It was shown that in a certain special case there exists an averaged temperature field τ_o defined by (18) and the modified temperature field given by (20) for which the evolution in time of the temperature field is described by a hyperbolic-type equation (20).

References

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