AN EFFECTIVE TOROIDAL APPROXIMATION FOR FREE LIQUID SURFACE BETWEEN ELLIPSOIDAL GRAINS

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Abstract. The capillary bridge between ellipsoidal grains is modelled within an effective toroidal approximation. We analyse the case when major semi axes of grains are parallel.

It is well known that mechanical properties of a granular material change strongly if some liquid is added [1-4]. Even small amount of liquid introduces the internal cohesion between grains resulting from capillary forces [5]. So, a growing scientific effort is devoted to understanding physical properties of wet granular materials. Most of grains are not spherical and then we have to analyze how the macroscopic curvature of contacted grains influences on an inter-grain adhesive energy. Here we consider the ellipsoidal grains. For this shape of grain there are three different grain-to-grain arrangements presented in Figure 1.

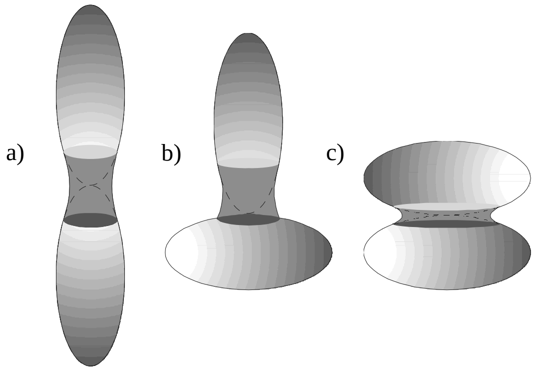

Fig. 1. Three different arrangements of ellipsoidal grains for saturated regime: a) A-A, b) A-B, c) B-B .

In our previous papers [6, 7] we analyzed the A-A arrangement. For that grains' orientation we were able to employed so-called toroidal approximation. Certainly, for A-B arrangement a toroidal approximation can not be used. However, for very low liquid volume we still can applied some version of such approximation for B-B grains' orientation. Thus, our objective in this paper is to compute the free liquid surface resulting from capillary bridges formed between two grains whose major semi-axes are parallel.

The geometry of the liquid bridge is characterized by wetting angle, surface tension of liquid, size and shape of grains. The curvature of the capillary bridge is described by the Laplace-Young equation (1)

$$\frac{\Delta P}{\gamma} = \frac{\frac{d^2 y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} - \frac{1}{y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}} \tag{1}$$

where: ΔP is the capillary suction pressure, γ is the liquid surface tension and unknown function $y(x)$ is a curvature of the capillary bridge. The Equation (1) cannot be solved analytically in general [3].

In case of $\Delta P = 0$, the Equation (1) yields a toroidal capillary surface with a given radius r for fixed wetting angle θ. The aim of this paper is to calculate a free liquid surface when the capillary suction pressure is almost equal zero.

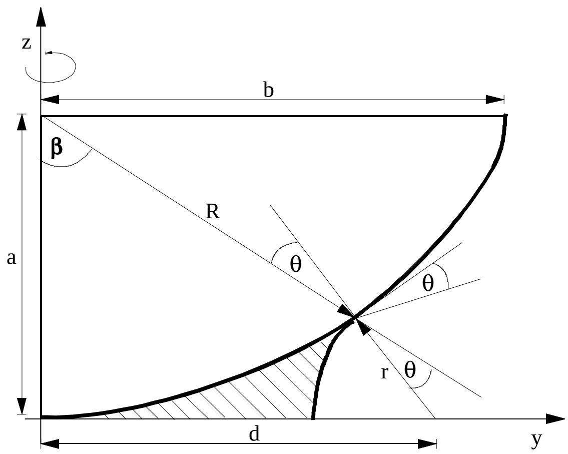

Fig. 2. The fragment of a cross-section of ellipsoidal grain and liquid bridge for the B-B arrangement. The shaded area *A* is filled with wetting liquid

The condition of the contact of two ellipsoidal grains (Fig. 2) is given by

$$\mathrm{tg}(\beta - \theta) = \frac{1}{k^2}\,\mathrm{tg}\beta \tag{2}$$

The geometry of the problem is characterized by a wetting angle θ, a material angle β and two semi-axes: the minor axis a and the major axis $b = ka,\ k > 1$.

From Figure 2 it follows dependence (3)

$$a = r\cos(\beta - \theta) + R\cos\beta \tag{3}$$

Conditions (2) and (3) lead to the implicit equality

$$\frac{s\,k^2}{\sqrt{k^4 + \mathrm{tg}^2\beta}} + \frac{k}{\sqrt{k^2 + \mathrm{tg}^2\beta}} = 1 \tag{4}$$

where $s = r/a$. For small values of s the Equation (4) can be written as

$$\mathrm{tg}^2\beta = 2k^2 s + (3k^2 - 2)s^2 + o(s)^3 \tag{5}$$

Since

$$d = r\sin(\beta - \theta) + R\sin\beta \tag{6}$$

thus from the Equation (4) it is gotten

$$d = a\,\mathrm{tg}\beta\left[\frac{1}{k^2} + \frac{k^2 - 1}{k\sqrt{k^2 + \mathrm{tg}^2\beta}}\right] \tag{7}$$

and taking Equation (5) into consideration it has been obtained

$$d \cong a\sqrt{2sk}\left[\left(1 - \frac{1}{4}s - \frac{33}{32}s^2\right) + \left(-\frac{1}{2}s + \frac{15}{8}s^2\right)\frac{1}{k^2} - \frac{1}{8}s^2\frac{1}{k^4}\right] + o(s)^3 \tag{8}$$

Now we are able to calculate the free liquid surface S. It is convenient to employ a spherical coordinates (ρ, α, φ) (Fig. 3).

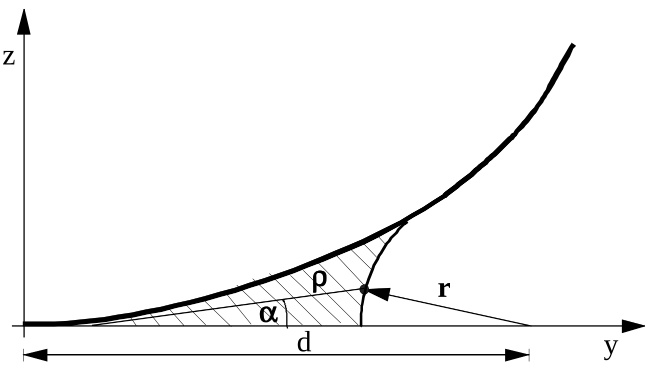

Fig. 3. Same as in Fig. 2. Parameters (ρ, α, φ) are the spherical coordinates used in Eqs. (9)-(16)

It has been obtained

$$|S| = \int_S dS = 8\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\alpha_o(\varphi)} \rho \sqrt{\rho^2 \cos^2\alpha + \left(\frac{\partial\rho}{\partial\alpha}\right)^2 \cos^2\alpha + \left(\frac{\partial\rho}{\partial\varphi}\right)^2}\, d\alpha \qquad (9)$$

where ρ denotes

$$\rho = d\cos\alpha - \sqrt{r^2 - d^2\sin^2\alpha},\ \ d = d(\varphi) \qquad (10)$$

as well as

$$\sin\alpha_o = \frac{1 - \dfrac{k}{\sqrt{k^2 + \mathrm{tg}^2\beta}}}{\sqrt{1 + \dfrac{k^2(1 + tg^2\beta)}{k^2 + \mathrm{tg}^2\beta} - \dfrac{2k}{\sqrt{k^2 + \mathrm{tg}^2\beta}}}} \qquad (11)$$

Hence after easy transformations have been carrying out

$$|S| = 8r\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\alpha_o(\varphi)} \frac{\rho^2\cos\alpha}{\sqrt{r^2 - d^2\sin^2\alpha}} \sqrt{1 + \left(\frac{\partial d}{\partial\varphi}\right)^2 \frac{1}{\cos^2\alpha}\left(\frac{1}{\rho^2} - \frac{1}{r^2}\sin^2\alpha\right)}\, d\alpha \qquad (12)$$

and after an expansion of second factor on the right side of the Equation (12) with respect to $\sin^2\alpha$, we have obtained

$$|S| = 8r^2\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\alpha_o(\varphi)} \frac{\left(\delta\cos\alpha - \sqrt{1 - \delta^2\sin^2\alpha}\right)^2}{\sqrt{1 - \delta^2\sin^2\alpha}} \cdot$$

$$\cdot\left[\sqrt{1 + \left(\frac{\partial\delta}{\partial\varphi}\right)^2 \frac{1}{(\delta - 1)^2}} - \frac{\delta}{2(\delta - 1)} \frac{\left(\dfrac{\partial\delta}{\partial\varphi}\right)^2}{\sqrt{1 + \left(\dfrac{\partial\delta}{\partial\varphi}\right)^2 \dfrac{1}{(\delta - 1)^2}}} \sin^2\alpha + \ldots\right] d\alpha \qquad (13)$$

where $\delta = \dfrac{d}{r}$.

Therefore, a lateral surface S of the liquid bridge is given by

$$|S| = 8r^2 \int_0^{\frac{\pi}{2}} [\sqrt{1+\left(\frac{\partial\delta}{\partial\varphi}\right)^2 \frac{1}{(\delta-1)^2}} \int_0^{\alpha_o(\varphi)} h(\delta,\alpha)\cdot d\alpha$$

$$-\frac{1}{2}\frac{\delta}{\delta-1}\frac{\left(\frac{\partial\delta}{\partial\varphi}\right)^2}{\sqrt{1+\left(\frac{\partial\delta}{\partial\varphi}\right)^2 \frac{1}{(\delta-1)^2}}} \int_0^{\alpha_o(\varphi)} h(\delta,\alpha)\sin^2\alpha\cdot d\alpha + \ldots]d\varphi \tag{14}$$

where:

$$h(\delta,\alpha) = \frac{\left(\delta\cos\alpha - \sqrt{1-\delta^2\sin^2\alpha}\right)^2 \cos\alpha}{\sqrt{1-\delta^2\sin^2\alpha}},$$

$$\delta = \delta(\varphi) = s\sqrt{2s}\left[\left(1-\frac{1}{4}s-\frac{33}{32}s^2\right)k(\varphi)+\left(-\frac{1}{2}s+\frac{15}{8}s^2\right)\frac{1}{k(\varphi)}-\frac{1}{8}s^2\frac{1}{k^3(\varphi)}+\ldots\right] \tag{15}$$

$$k(\varphi) = \frac{k^2}{\sin^2\varphi + k^2\cos^2\varphi}, \quad k>1,\ 0\le\varphi\le\frac{\pi}{2} \tag{16}$$

In accordance with the Equation (8) which was used after a rotation with an angle φ. At the above-mentioned dependence it is shown, integrals with respect to α are infinitesimal ones, while with respect to φ (including the expression $\sqrt{1+\left(\frac{\partial\delta}{\partial\varphi}\right)^2 \frac{1}{(\delta-1)^2}}$), and they lead to elliptic integrals.

The cohesive energy E for the B-B grain arrangement (see Figure 1) is equal to $E=\gamma S$, S is given by the Equation (14).

In conclusion we computed the cohesive energy between ellipsoidal grains due to the small amount of liquid. We approximate free - liquid surface of the bridge by a curve linear toroid for B-B arrangement, see Figure 1. It allow us to study the grain - grain reorientation which can be induced fluctuation of humidity and it will be analyzed.

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