FREE VIBRATIONS OF A SYSTEM OF NON-UNIFORM BEAMS COUPLED BY ELASTIC LAYERS

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Abstract. The paper concerns a free vibration problem of a system of non-uniform beams coupled by non-homogeneous elastic layers. To solve the boundary problem, the Green's function method was applied. An example of the Green's function corresponding to the differential operator with variable coefficients is presented. The frequency equation is obtained by using a quadrature rule of a Newton-Cotes type.

Introduction

The vibration of systems composed of uniform beams coupled by translational springs or elastic layers has been studied extensively in the literature (see for example [1-5]). The authors of paper [1] deal with the problem of the natural transverse vibrations of a system consisting of two clamped-free beams to which several double spring-mass systems are attached. The solution of similar vibration problem which contains possible combinations of classical boundary conditions, but the beams are connected by many translational springs without masses, was presented in paper [2]. Free vibration of a system of many beams connected by translational springs was studied in reference [3]. The papers [4, 5] are devoted to the vibrations of uniform beams connected by homogeneous elastic layer. Vibration of two uniform plates coupled by non-homogeneous layer was investigated in paper [6]. The solutions of problems presented in these papers are determined by using an exact methods. In references [1-3] and [7, 8] the Green's function method was applied. Derivation of the Green's functions for uniform beams is presented in reference [3]. The Green's functions for any cases of non-uniform beams were determined in papers [7, 8].

The purpose of this paper is solution of the free vibration problem of a combined system consisting of non-uniform beams coupled by non-homogeneous elastic layers. The solution of the problem is obtained by using a Green's function which corresponds to a differential operator occurring in differential description of the beam vibration. The Green's function is expressed by Bessel functions of the first and second kind. In order to derivation of the eigenequation and eigenfunctions a quadrature method is applied.

1. Formulation of the problem

Consider a system of n beams of length L which are coupled by n-1 elastic layers with the stiffness module $k_j(x)$ (j = 1,..., n-1). A sketch of the system considered is presented in Figure 1.

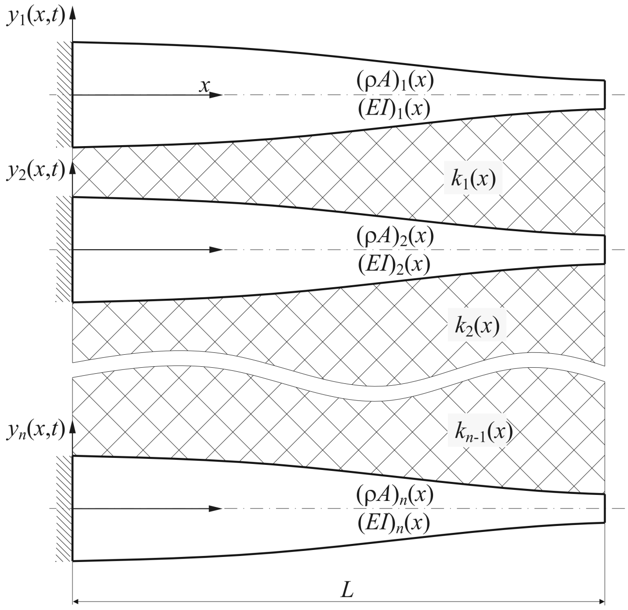

Fig. 1. A sketch of the system of n non-uniform beams coupled by n-1 elastic layers

Let us assume that the non-uniform beams are characterized by the functions $(\rho A)_i(x)$ and $(EI)_i(x)$ which denote the mass per unit length and flexural rigidity of the i-th beam at arbitrary co-ordinate x, respectively. The differential equations of lateral vibrations of the beams are:

$$\Lambda_1[y_1(x,t)] = k_1(x)[y_2(x,t) - y_1(x,t)] \tag{1a}$$

$$\Lambda_i[y_i(x,t)] = k_{i-1}(x)[y_{i-1}(x,t) - y_i(x,t)] + k_i(x)[y_{i+1}(x,t) - y_i(x,t)]$$
$$i = 2, 3, \ldots, n-1 \tag{1b}$$

$$\Lambda_n[y_n(x,t)] = k_{n-1}(x)[y_{n-1}(x,t) - y_n(x,t)] \tag{1c}$$

where $x \in [0, L]$ and Λ_i (i = 1, 2,…, n) are differential operators in the form:

$$\Lambda_i \equiv \frac{\partial^2}{\partial x^2}\left[(EI)_i(x)\frac{\partial^2}{\partial x^2}\right] + (\rho A)_i(x)\frac{\partial^2}{\partial t^2} \tag{2}$$

Each function $y_i(x,t)$ satisfy homogeneous boundary conditions which correspond to the attachments of ends of the beams. The conditions can be written symbolically in the form:

$$\mathrm{B}_{0i}\left[y_i(x,t)\right]\Big|_{x=0}=0\,,\quad \mathrm{B}_{1i}\left[y_i(x,t)\right]\Big|_{x=L}=0 \qquad i = 1, 2,\ldots, n \tag{3}$$

In order to consider free harmonic motion of the beams with frequency ω, the beam deflections are assumed in the form $y_i(x,t)=W_i(x)\cos\omega t$. By substituting these equations into equations (1a-c) and boundary conditions (3), one obtains:

$$\frac{d^2}{dx^2}\left[(EI)_1(x)\frac{d^2W_1(x)}{dx^2}\right]-\omega^2(\rho A)_1(x)W_1(x)=k_1(x)\left[W_2(x)-W_1(x)\right] \tag{4a}$$

$$\frac{d^2}{dx^2}\left[(EI)_i(x)\frac{d^2W_i(x)}{dx^2}\right]-\omega^2(\rho A)_i(x)W_i(x)=$$
$$=k_{i-1}(x)\left[W_{i-1}(x)-W_i(x)\right]+k_i(x)\left[W_{i+1}(x)-W_i(x)\right]$$
$$i = 2, 3,\ldots, n\text{-}1 \tag{4b}$$

$$\frac{d^2}{dx^2}\left[(EI)_n(x)\frac{d^2W_n(x)}{dx^2}\right]-\omega^2(\rho A)_n(x)W_n(x)=k_{n-1}(x)\left[W_{n-1}(x)-W_n(x)\right] \tag{4c}$$

$$\mathbf{B}_{0i}\left[W_i(x)\right]\Big|_{x=0}=0\,,\qquad \mathbf{B}_{1i}\left[W_i(x)\right]\Big|_{x=L}=0 \tag{5}$$

Introducing the non-dimensional co-ordinates $\xi=x/L$, $W_i(\xi)=W_i(x)/L$ and quantities: $(EI)_i(\xi)=(EI)_i(x)/(EI)_i(0)$, $(\rho A)_i(\xi)=(\rho A)_i(x)/(\rho A)_i(0)$, $\mu_i=(EI)_i(0)/(EI)_{i+1}(0)$, $\Omega_i^4=\omega^2(\rho A)_i(0)L^4/(EI)_i(0)$, $K_i(\xi)=k_i(\xi)L^3/(EI)_i(0)$, the equations (4)-(5) can be written in the following form:

$$\tilde{L}_1\left[W_1(\xi)\right]=K_1(\xi)\left[W_2(\xi)-W_1(\xi)\right] \tag{6a}$$

$$\tilde{L}_i\left[W_i(\xi)\right]=\mu_{i-1}K_{i-1}(\xi)\left[W_{i-1}(\xi)-W_i(\xi)\right]+K_i(\xi)\left[W_{i+1}(\xi)-W_i(\xi)\right],$$
$$i = 2,\ldots, n\text{-}1 \tag{6b}$$

$$\tilde{L}_n\left[W_n(\xi)\right]=\mu_{n-1}K_{n-1}(\xi)\left[W_{n-1}(\xi)-W_n(\xi)\right] \tag{6c}$$

$$\mathbf{B}_{0i}\left[W_i(\xi)\right]\Big|_{\xi=0}=0\,,\quad \mathbf{B}_{1i}\left[W_i(\xi)\right]\Big|_{\xi=1}=0 \tag{7}$$

Differential operators $\tilde{\Lambda}_i$ are as follows:

$$\tilde{\Lambda}_i \equiv \frac{d^2}{d\xi^2}\left[(EI)_i(\xi)\frac{d^2}{d\xi^2}\right] - \Omega_i^4(\rho A)_i(\xi) \tag{8}$$

The solution of the eigenproblem (6)-(7) is obtained by using the Green's functions of the operators (8). This approach leads to an integral problem which consists in finding the eigenfrequency ω, for which there exists a non-trivial solution of the system of Fredholm equations of the second kind. These equations are obtained by multiply both side of equations (6) by the Green's functions $G_i(\xi,\eta)$ and than integrate over [0,1]. Using boundary conditions one obtains:

$$W_1(\xi) = \int_0^1 K_1(\eta)\left[W_2(\eta) - W_1(\eta)\right]G_1(\xi,\eta)\,d\eta \tag{9a}$$

$$W_i(\xi) = \int_0^1 \left\{\mu_{i-1}K_{i-1}(\eta)\left[W_{i-1}(\eta) - W_i(\eta)\right] + K_i(\eta)\left[W_{i+1}(\eta) - W_i(\eta)\right]\right\}G_i(\xi,\eta)\,d\eta$$

$$i = 2, 3, \ldots, n\text{-}1 \tag{9b}$$

$$W_n(\xi) = \mu_{n-1}\int_0^1 K_{n-1}(\eta)\left[W_{n-1}(\eta) - W_n(\eta)\right]G_n(\xi,\eta)\,d\eta \tag{9c}$$

By subtraction both sides of i-th and (i+1)-th equations, for i = 1,..., n-1, the following equations are obtained:

$$\bar{W}_i(\xi) = \int_0^1 K_{i+1}(\eta)\bar{W}_{i+1}(\eta)G_{i+1}(\xi,\eta)\,d\eta - \int_0^1 K_i(\eta)\bar{W}_i(\eta)\left[G_{i+1}(\xi,\eta)\mu_i + G_i(\xi,\eta)\right]d\eta$$

$$+\int_0^1 \mu_{i-1}K_{i-1}(\eta)\bar{W}_{i-1}(\eta)G_i(\xi,\eta)\,d\eta \qquad i = 1,\ldots, n\text{-}1 \tag{10}$$

where $\bar{W}_i(\xi) = W_{i+1}(\xi) - W_i(\xi)$ and $K_0(\eta) = K_n(\eta) = 0$ for $\eta \in [0,1]$.

In order to determine the eigenfrequencies ω, to derivation of the integrals occurring in equation (10) an approximate method should be used. In this paper a quadrature rule of a Newton-Cotes type is proposed [9]. In this case equations (10) follows that:

$$\bar{W}_i(\xi) = \sum_{j=1}^{m} K_{i+1}(\theta_j)\bar{W}_{i+1}(\theta_j)G_{i+1}(\xi,\theta_j)w_j$$

$$-\sum_{j=1}^{m}\left\{K_i(\theta_j)\bar{W}_i(\theta_j)\left[G_{i+1}(\xi,\theta_j)\mu_i + G_i(\xi,\theta_j)\right] + \mu_{i-1}K_{i-1}(\theta_j)\bar{W}_{i-1}(\theta_j)G_{i-1}(\xi,\theta_j)\right\}w_j \tag{11}$$

where w_j are the weighting coefficients and θ_j (j = 1, 2,…, m) are the knots of the quadrature.

Substituting, successively, $\xi = \theta_p$ (p = 1, 2,…, m) into equation (11) for i = 1, 2,…, n-1, a set of m(n-1) linear equations with unknown $\bar{W}_i\left(\theta_j\right)$ is obtained. This equation system can be written in the form:

$$\boldsymbol{A}\bar{\boldsymbol{W}} = \boldsymbol{0} \tag{12}$$

where

$$\bar{\boldsymbol{W}} = \left[\bar{W}_1(\theta_1)\,\bar{W}_1(\theta_2)...\bar{W}_1(\theta_m)\bar{W}_2(\theta_1)...\bar{W}_2(\theta_m)...\bar{W}_{n-1}(\theta_1)\bar{W}_{n-1}(\theta_2)...\bar{W}_{n-1}(\theta_m)\right]^T$$

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{O} & \mathbf{O} & \cdots & & & \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \mathbf{B}_{23} & \mathbf{O} & \cdots & & \mathbf{O} & \\ \mathbf{O} & \mathbf{B}_{32} & \mathbf{B}_{33} & \mathbf{B}_{34} & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & \mathbf{O} & & \cdots & \mathbf{B}_{n-2,n-3} & \mathbf{B}_{n-2,n-2} & \mathbf{B}_{n-2,n-1} \\ & & & & \cdots & \mathbf{O} & \mathbf{B}_{n-1,n-2} & \mathbf{B}_{n-1,n-1} \end{bmatrix}$$

and

$$\mathbf{B}_{i,i-1} = \left[b_{jp}^{(i)}\right]_{1\le j,p\le m}, \quad \mathbf{B}_{i,i} = \left[\bar{b}_{jp}^{(i)}\right]_{1\le j,p\le m}, \quad \mathbf{B}_{i,i-1} = \left[\bar{\bar{b}}_{jp}^{(i)}\right]_{1\le j,p\le m}$$

where
$$b_{jp}^{(i)} = \mu_{i-1}K_{i-1}\left(\theta_j\right)G_{i-1}\left(\theta_j,\theta_p\right)w_j$$

$$\bar{b}_{jp}^{(i)} = -\delta_{jp} - K_i\left(\theta_j\right)\left[\mu_i G_{i+1}\left(\theta_j,\theta_p\right) + G_i\left(\theta_j,\theta_p\right)\right]w_j$$

$$\bar{\bar{b}}_{jp}^{(i)} = K_{i+1}\left(\theta_j\right)G_{i+1}\left(\theta_j,\theta_p\right)w_j$$

For a non-trivial solution of the equation (12) determinant of the coefficient matrix $\boldsymbol{A}$ is set equal to zero, yielding the frequency equation:

$$\det \boldsymbol{A} = 0 \tag{13}$$

The equation (13), is then solved numerically with respect to eigenfrequencies. The eigenfunctions corresponding to the eigenfrequencies are derived by using equations (11).

2. Example of Green's function corresponding to non-uniform beam

Let us consider a system of two non-uniform beams connected by elastic layer with a constant stiffness k. Assume also that:

$$(EI)_i(x) = (EI)_i(0)\left(\frac{\alpha-1}{L}x+1\right)^4, \quad (\rho A)_i(x) = (\rho A)_i(0)\left(\frac{\alpha-1}{L}x+1\right)^2$$

$$i = 1, 2 \tag{14}$$

$\alpha = b_{iL}/b_{i0} = h_{iL}/h_{i0}$ ($\alpha \neq 1$), where b_{i0}, h_{i0} and b_{iL}, h_{iL} are widht and height of the i-th beam at points $x = 0$ and $x = L$, respectively. Non-dimensional co-ordinates are assumed as follows: $\xi = \frac{\alpha-1}{L}x+1$. In this case $\xi \in [1,\alpha]$. Differential operators $\tilde{\Lambda}_i$ occuring on the left side of equations (6a-c) are:

$$\tilde{\Lambda}_i \equiv \frac{d^2}{d\xi^2}\left[\xi^4\frac{d^2}{d\xi^2}\right] - \left(\frac{\beta_i}{2}\right)^4\xi^2, \quad i = 1, 2 \tag{15}$$

where: $\frac{\beta_i}{2} = \Omega_i$, $\Omega_i^4 = \frac{\omega^2(\rho A)_i(0)L^4}{(\alpha-1)^4(EI)_i(0)}$ and boundary conditions (7) have the form:

$$\mathbf{B}_{0i}\left[W_i(\xi)\right]\Big|_{\xi=1} = 0, \quad \mathbf{B}_{1i}\left[W_i(\xi)\right]\Big|_{\xi=\alpha} = 0 \tag{16}$$

The Green's functions of the differential operators (15) can be written in the form [5]:

$$G_i(\xi,\eta) = \frac{1}{\xi}\left[C_{1i}J_2\left(\beta_i\sqrt{\xi}\right) + C_{2i}Y_2\left(\beta_i\sqrt{\xi}\right) + C_{3i}I_2\left(\beta_i\sqrt{\xi}\right) + C_{4i}K_2\left(\beta_i\sqrt{\xi}\right)\right] + G_{1i}(\xi,\eta)H(\xi-\eta) \tag{17}$$

where $H(\ )$ is the Heavyside function, J_v, Y_v, I_v, K_v are the Bessel functions of the first and second kind and G_{1i} are functions which can be expressed as:

$$\begin{aligned} G_{1i}(\xi,\eta) = \frac{4}{\beta^2\eta\xi}\Big\{ & I_2\left(\beta_i\sqrt{\xi}\right)K_2\left(\beta_i\sqrt{\eta}\right) - I_2\left(\beta_i\sqrt{\eta}\right)K_2\left(\beta_i\sqrt{\xi}\right) + \\ & + \frac{\pi}{2}\left[J_2\left(\beta_i\sqrt{\xi}\right)Y_2\left(\beta_i\sqrt{\eta}\right) - J_2\left(\beta_i\sqrt{\eta}\right)Y_2\left(\beta_i\sqrt{\xi}\right)\right]\Big\} \end{aligned} \tag{18}$$

The coefficients C_{1i}, C_{2i}, C_{3i}, C_{4i} ($i = 1, 2$), occurring in (17) are determined on basis of the boundary conditions. Assuming that the cantilever beam is considered, the coefficients C_{3i} and C_{4i} can be determined from the conditions: $G\big|_{\xi=1} = G'_{\xi}\big|_{\xi=1} = 0$. This coefficients are:

$$C_{3i} = C_{1i}\phi_1(\beta_i) + C_{2i}\phi_2(\beta_i)\,,\, C_{4i} = C_{1i}\phi_3(\beta_i) + C_{2i}\phi_4(\beta_i) \tag{19}$$

where:

$$\phi_1(\beta_i) = -\beta_i\left[K_3(\beta)J_2(\beta_i) - K_2(\beta_i)J_3(\beta_i)\right]$$

$$\phi_2(\beta_i) = -\beta_i\left[K_3(\beta_i)Y_2(\beta_i) - K_2(\beta_i)Y_3(\beta_i)\right]$$

$$\phi_3(\beta_i) = -\beta_i\left[I_3(\beta_i)J_2(\beta_i) - I_2(\beta_i)J_3(\beta_i)\right]$$

$$\phi_4(\beta_i) = -\beta_i\left[I_3(\beta_i)Y_2(\beta_i) - I_2(\beta_i)Y_3(\beta_i)\right] \tag{20}$$

The coefficients C_{1i}, C_{2i} are determined from the conditions: $G''_{\xi\xi}\big|_{\xi=\alpha} = 0$, $G'''_{\xi\xi\xi}\big|_{\xi=\alpha} = 0$.

The Green's functions for beams characterized by other functions describing variable cross-sections are given in reference [6].

Conclusions

In this paper an exact solution to the problem of free vibration of a system of non-uniform beams coupled by non-homogeneous elastic layers is presented. The formulation of the problem establish the differential equations of motion of the beams and boundary conditions corresponding to the attachments of the beam ends. By using properties of Green's functions an integral formulation of the problem was achieved. The frequency equation of the beam system was obtained by application of a quadrature method to the integral equations. The presented solution can be used in numerical investigation of vibration of the considered systems.

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