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APPLICATION OF FDM FOR NUMERICAL SOLUTION OF HYPERBOLIC HEAT CONDUCTION EQUATION

***Ewa Majchrzak*[1,2], *Jerzy Mendakiewicz*[1]**

[1] *Silesian University of Technology, Gliwice, Poland*
[2] *Institute of Mathematics and Computer Science, Czestochowa University of Technology, Poland*

Abstract. In the paper the algorithm of numerical solution of hyperbolic heat conduction equation is presented. The explicit and implicit variants of finite differences method are applied and the results of computations are shown.

1. Hyperbolic heat conduction

Hyperbolic heat conduction equation is of the form [1, 2]

$$c\rho\left[\tau\frac{\partial^2 T(x,t)}{\partial t^2}+\frac{\partial T(x,t)}{\partial t}\right]=\nabla\left[\lambda\nabla T(x,t)\right]+Q(x,t)+\tau\frac{\partial Q(x,t)}{\partial t} \tag{1}$$

where c [J/(kgK)] is the specific heat, ρ [kg/m^3] is the density, λ [W/(mK)] is the thermal conductivity, $Q(x, t)$ [W/m^3] is the source function, τ [s] is the relaxation time, T, x, t denote temperature, spatial co-ordinates and time.

For constant values λ, c, ρ the equation (1) can be written as follows

$$\tau\frac{\partial^2 T(x,t)}{\partial t^2}+\frac{\partial T(x,t)}{\partial t}=a\nabla^2 T(x,t)+\frac{Q(x,t)}{c\rho}+\frac{\tau}{c\rho}\frac{\partial Q(x,t)}{\partial t} \tag{2}$$

where $a = \lambda/(c\rho)$ is the thermal diffusivity.

Equation (2) is supplemented by boundary condition

$$x\in\Gamma: \quad T(x,t)=T_b \tag{3}$$

and initial conditions

$$t=0: \quad \begin{cases} T(x,t)=T_0 \\ \dfrac{\partial T(x,t)}{\partial t}=0 \end{cases} \tag{4}$$

2. Approximation of time derivatives

In order to solve the problem (2), (3), (4) the time grid

$$0 = t^0 < t^1 < \ldots < t^{f-2} < t^{f-1} < t^f < \ldots < t^F < \infty \tag{5}$$

with constant step $\Delta t = t^f - t^{f-1}$ is introduced.

Using the Lagrange interpolation [3] for the points $\left(t^{f-2}, T^{f-2}\right)$, $\left(t^{f-1}, T^{f-1}\right)$, $\left(t^f, T^f\right)$, where $T^{f-2} = T\left(x, t^{f-2}\right)$, $T^{f-1} = T\left(x, t^{f-1}\right)$, $T^f = T\left(x, t^f\right)$ one obtains

$$\begin{gathered} T(x,t) = T^{f-2} \frac{\left(t - t^{f-1}\right)\left(t - t^f\right)}{\left(t^{f-2} - t^{f-1}\right)\left(t^{f-2} - t^f\right)} + T^{f-1} \frac{\left(t - t^{f-2}\right)\left(t - t^f\right)}{\left(t^{f-1} - t^{f-2}\right)\left(t^{f-1} - t^f\right)} + \\ + T^f \frac{\left(t - t^{f-2}\right)\left(t - t^{f-1}\right)}{\left(t^f - t^{f-2}\right)\left(t^f - t^{f-1}\right)} \end{gathered} \tag{6}$$

or

$$T(x,t) = T^{f-2} \frac{\left(t - t^{f-1}\right)\left(t - t^f\right)}{2\left(\Delta t\right)^2} - T^{f-1} \frac{\left(t - t^{f-2}\right)\left(t - t^f\right)}{\left(\Delta t\right)^2} + T^f \frac{\left(t - t^{f-2}\right)\left(t - t^{f-1}\right)}{2\left(\Delta t\right)^2} \tag{7}$$

On the basis of (7) the time derivatives are calculated and then

$$\left.\frac{\partial T(x,t)}{\partial t}\right|_{t = t^f} = \frac{T^{f-2} - 4T^{f-1} + 3T^f}{2\,\Delta t} \tag{8}$$

while

$$\frac{\partial^2 T(x,t)}{\partial t^2} = \frac{T^{f-2} - 2T^{f-1} + T^f}{\left(\Delta t\right)^2} \tag{9}$$

Taking into account the formulas (8), (9) the following approximation of left-hand side L of equation (2) is obtained

$$\begin{gathered} L = \tau \frac{T(x, t^{f-2}) - 2T(x, t^{f-1}) + T(x, t^f)}{\left(\Delta t\right)^2} + \\ + \frac{T(x, t^{f-2}) - 4T(x, t^{f-1}) + 3T(x, t^f)}{2\,\Delta t} \end{gathered} \tag{10}$$

It should be pointed out that the right-hand side R of equation (2) can be written for time $t = t^{f-2}$, for time $t = t^{f-1}$ or for time $t = t^f$.

3. Finite differences method

For 2D problem and domain oriented in Cartesian co-ordinate system $x = \{x, y\}$ one has

$$\nabla^2 T(x, y, t) = \frac{\partial^2 T(x, y, t)}{\partial x^2} + \frac{\partial^2 T(x, y, t)}{\partial y^2} \tag{11}$$

The following approximation of (11) with respect to the geometrical co-ordinates for constant mesh step h can be taken into account [3, 4]

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)_{i,j} = \frac{1}{h^2}\left(T_{i-1,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1} - 4T_{i,j}\right) \tag{12}$$

where: $T_{i,j} = T(x_i, y_i, t)$, $T_{i-1,j} = T(x_{i-1}, y_{i-1}, t)$ etc.
We assume that $Q(x, y, t) = 0$. Using the explicit scheme of FDM one obtains the following approximation of equation (2)

$$\tau\frac{T_{i,j}^{f-2} - 2T_{i,j}^{f-1} + T_{i,j}^{f}}{(\Delta t)^2} + \frac{T_{i,j}^{f-2} - 4T_{i,j}^{f-1} + 3T_{i,j}^{f}}{2\,\Delta t} = \frac{a}{h^2}\left(T_{i-1,j}^{s} + T_{i+1,j}^{s} + T_{i,j-1}^{s} + T_{i,j+1}^{s} - 4T_{i,j}^{s}\right) \tag{13}$$

where $s = f - 2$ or $s = f - 1$, while for the implicit scheme of the FDM one has

$$\tau\frac{T_{i,j}^{f-2} - 2T_{i,j}^{f-1} + T_{i,j}^{f}}{(\Delta t)^2} + \frac{T_{i,j}^{f-2} - 4T_{i,j}^{f-1} + 3T_{i,j}^{f}}{2\,\Delta t} = \frac{a}{h^2}\left(T_{i-1,j}^{f} + T_{i+1,j}^{f} + T_{i,j-1}^{f} + T_{i,j+1}^{f} - 4T_{i,j}^{f}\right) \tag{14}$$

From equation (13) results that

$$\left[\frac{\tau}{(\Delta t)^2} + \frac{3}{2\Delta t}\right] T_{i,j}^{f} = \frac{a}{h^2}\left(T_{i-1,j}^{s} + T_{i+1,j}^{s} + T_{i,j-1}^{s} + T_{i,j+1}^{s} - 4T_{i,j}^{s}\right) - \left[\frac{\tau}{(\Delta t)^2}\left(T_{i,j}^{f-2} - 2T_{i,j}^{f-1}\right) + \frac{1}{2\Delta t}\left(T_{i,j}^{f-2} - 4T_{i,j}^{f-1}\right)\right] \tag{15}$$

this means

$$T_{i,j}^{f} = \frac{2a(\Delta t)^2}{h^2(2\tau + 3\Delta t)}\left(T_{i-1,j}^{s} + T_{i+1,j}^{s} + T_{i,j-1}^{s} + T_{i,j+1}^{s} - 4T_{i,j}^{s}\right) - \frac{2\tau}{2\tau + 3\Delta t}\left(T_{i,j}^{f-2} - 2T_{i,j}^{f-1}\right) - \frac{\Delta t}{2\tau + 3\Delta t}\left(T_{i,j}^{f-2} - 4T_{i,j}^{f-1}\right) \tag{16}$$

From equation (14) one obtains

$$\left[\frac{\tau}{(\Delta t)^2} + \frac{3}{2\Delta t} + \frac{4a}{h^2}\right]T_{i,j}^{f} - \frac{a}{h^2}\left(T_{i-1,j}^{f} + T_{i+1,j}^{f} + T_{i,j-1}^{f} + T_{i,j+1}^{f}\right) = \left[\frac{\tau}{(\Delta t)^2}\left(T_{i,j}^{f-2} - 2T_{i,j}^{f-1}\right) + \frac{1}{2\Delta t}\left(T_{i,j}^{f-2} - 4T_{i,j}^{f-1}\right)\right] \tag{17}$$

Using formula (16) the stability criterion should be formulated [3, 4].

4. Results of computations

The rectangular domain of dimensions 0.02 × 0.04 m has been considered. The following thermophysical parameters have been assumed: λ = 1 W/(mK), c = 4000 J/(kgK), ρ = 1000 kg/m^3. The initial temperature equals T_0 = 50°C. On the left side of rectangle the boundary condition in the form

$$0 \le y \le 0.04: \quad T(0,\, y,\, t) = \frac{1}{L^2}\left[(2y - L)^2\, T_0 - 4y(y - L)\, T_{\max}\right] \tag{18}$$

is accepted (L = 0.04 m, T_0 = 50°C, $T_{\max}$ = 100°C), on the remaining part of the boundary the temperature T_b = 50°C is assumed. The problem has been solved using the explicit scheme of finite differences method (c.f. equation (16) for $s = f - 1$) with mesh step h = 0.001 m and time step Δt = 1 s. The computations have been done for relaxation time τ = 0 and τ = 15 s.

In Figure 1 the positions of isotherms after 60, 300 and 3600 s are shown. On the left hand-side the results for τ = 0 s are presented, on the right-hand side for τ = 15 s.

Figure 2 illustrates the courses of heating curves both for τ = 0 as well as for τ = 15 s at the internal points (0.005, 0.02), (0.01, 0.02) (center of rectangular) and (0.015, 0.02).

It is visible that the differences between the solutions for τ = 0 and τ = 15 s are especially big for the initial stages of heating process.

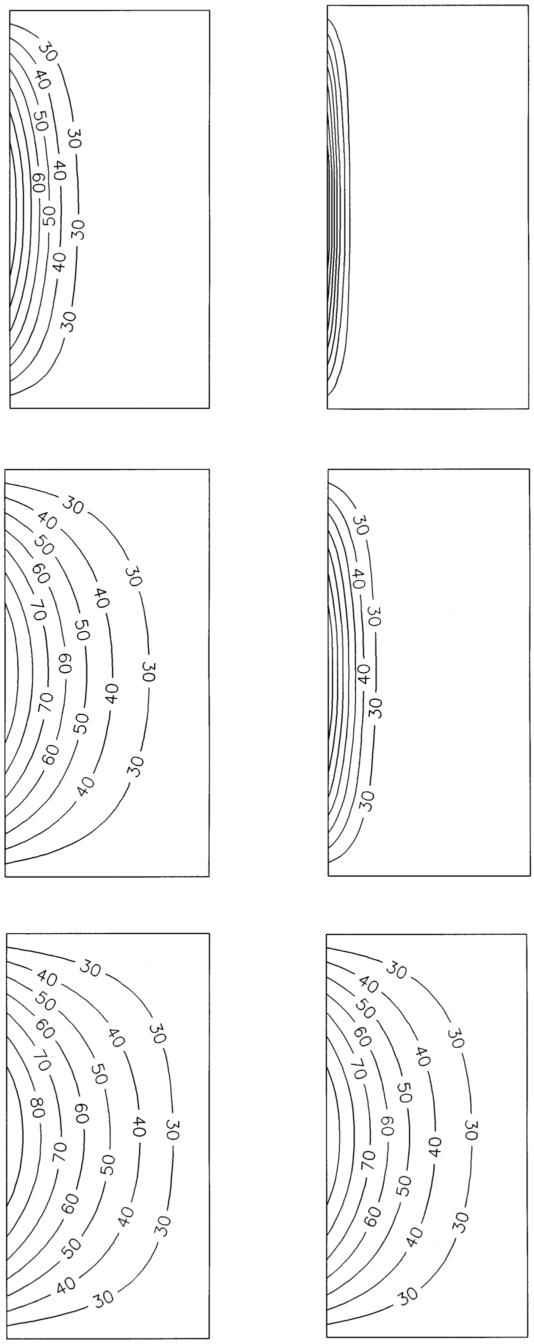

Fig. 1. Temperature distribution after 60, 300 and 3600 s (τ = 0 and τ = 15 s)

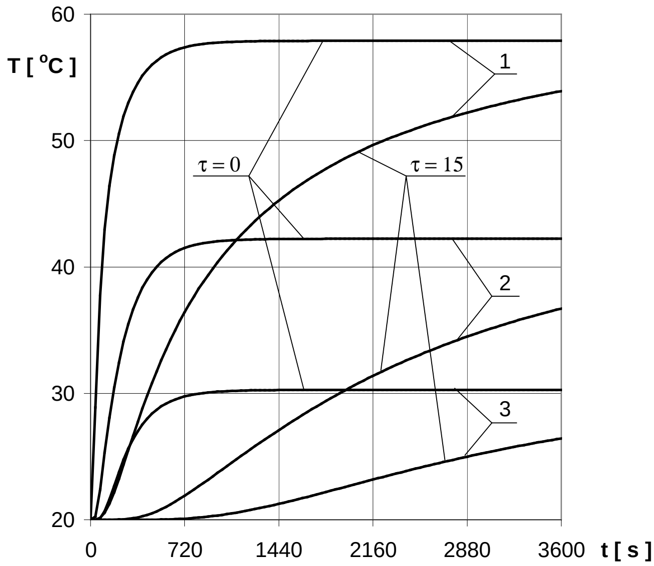

Fig. 2. Heating curves (1 - point (0.005, 0.02), 2 - point (0.01, 0.02), 3 - point (0.015, 0.02))

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