ANALYSIS OF OPEN QUEUEING NETWORK WITH DEPENDENT ON TIME PARAMETERS OF INPUT FLOW AND SERVICING

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Abstract. The multivariate generating functions method for obtaining of the open queueing network state probabilities, functioning with heavy load conditions, with dependent on time parameters of input flow and servicing is considered in this paper.

1. The system of Kolmogorov difference-differential equations for queueing network state probabilities

Let's consider open queueing network (QN), consisting of n queueing systems (QS), each of which has m_i identical service lines. We will examine the case, when parameters of input flow and servicing parameters depends on time, i.e. in the time interval $[t, t+\Delta t)$ message enters to the network with probability $\lambda(t)\Delta t + o(\Delta t)$, and if at the time moment t there is an message being servicing at the i-th QS then it's servicing will be finished in the time interval $[t, t+\Delta t)$ with the probability $\mu_i(t)\Delta t + o(\Delta t),\ \ i=\overline{1,n}$. With the probability p_{0i} message will come to peripheral system S_i from the outside, $i=\overline{1,n-1},\ \sum_{i=1}^{n-1} p_{0i}=1,$ after finished servicing in there this message will transfer to the central system S_n, and then will leave the network. Service disciplines in network systems are FIFO. Each system has unlimited number of waiting places. Service time in queueing system's lines have exponential distribution with various parameters for various systems. As a networks state at the moment of time t we will understand vector $k(t)=(k,t)=(k_1,k_2,\ldots,k_n,t)$, where k_i - number of messages in QS S_i, $i=\overline{1,n}$. The analysis of such network with central QS by multivariate generating functions method, when $\lambda(t)=\lambda$, $\mu_i(t)=\mu_i,\ \ i=\overline{1,n}$, was carried out in paper [1]. Let us designate through I_i - vector of dimension n with zero-components, except for i-th, which is equal to 1.

Lemma. *State probabilities of considered network satisfy to set of difference--differential equations:*

$$\frac{dP(k,t)}{dt} = -\left\{\lambda(t) + \sum_{i=1}^{n}\mu_i(t)\min(m_i,k_i)\right\}P(k,t) + \lambda(t)\sum_{i=1}^{n-1} p_{0i}u(k_i)P(k-I_i,t) +$$

$$+\sum_{i=1}^{n-1}\mu_i(t)u(k_n)\min(m_i,k_i+1)P(k+I_i-I_n,t) + \mu_n(t)\min(m_n,k_n+1)P(k+I_n,t) \quad (1)$$

Proof. In view of servicing time exponentiality the random process $k(t)=(k,t)$ is Markovian with countable states number. Following transitions to the state (k,t) during period of time Δt are possible:

a) from the state $(k+I_i-I_n,t)$ with probability

$$\mu_i(t)u(k_n)\min(m_i,k_i+1)\Delta t + o(\Delta t),\ \ i=\overline{1,n-1},$$

where $u(x)$ - Heaviside function;

b) from the state $(k-I_i,t)$ with probability $\lambda(t)p_{0i}u(k_i)\Delta t + o(\Delta t),\ \ i=\overline{1,n-1};$

c) from the state $(k+I_n,t)$ with probability $\mu_n(t)\min(m_n,k_n+1)\Delta t + o(\Delta t);$

d) from the state (k,t) with probability

$$1-\left[\lambda(t)+\sum_{i=1}^{n}\mu_i(t)\min(m_i,k_i)\right]\Delta t + o(\Delta t)$$

e) from other states with probability $o(\Delta t)$.

Then using composite probability function it is possible to write

$$P(k,t+\Delta t) = \sum_{i=1}^{n-1}\mu_i(t)u(k_n)\min(m_i,k_i+1)\Delta t\,P(k+I_i-I_n,t) + \lambda(t)\sum_{i=1}^{n-1}p_{0i}u(k_i)\Delta t\,P(k-I_i,t) +$$

$$+\mu_n(t)\min(m_n,k_n+1)\Delta t\,P(k+I_n,t) + \left\{1-\left[\lambda(t)+\sum_{i=1}^{n}\mu_i(t)\min(m_i,k_i)\right]\Delta t\right\}P(k,t) + o(\Delta t)$$

Dividing both left and right parts of this relation by Δt and passing to limit where $\Delta t \to 0,$ we will obtain a set of equations for network state probabilities (1).

2. Network state probabilities for the case *n* = 3

Let n = 3 and all queueing systems are single-line, i.e. $m_i = 1,\ i = \overline{1,n}.$ Let's suppose also, that all systems functioning in heavy load conditions, i.e. $k_i(t) > 0 \quad \forall t > 0, \quad i = \overline{1,n}$ (there are any servicing messages at any moment of time). Then set (1) in the lemma will look like

$$\frac{dP(k,t)}{dt} = -\left\{\lambda(t) + \sum_{i=1}^{3}\mu_i(t)\right\}P(k,t) + \lambda(t)\sum_{i=1}^{2} p_{0i}u(k_i)P(k - I_i, t) +$$

$$+\sum_{i=1}^{2}\mu_i(t)u(k_3)P(k + I_i - I_n, t) + \mu_3(t)P(k + I_3, t)$$

Let's define three-dimensional generating function. We will assume for $z = (z_1, z_2, z_3)$

$$\Psi_3(z,t) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} P(k_1,k_2,k_3,t)z_1^{k_1}z_2^{k_2}z_3^{k_3} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} P(k,t)z_1^{k_1}z_2^{k_2}z_3^{k_3} \qquad (2)$$

Multiplying each equation of set (1) by $z_1^{k_1}z_2^{k_2}z_3^{k_3}$ and summing up by all possible values k_1, k_2, k_3 we can obtain:

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{dP(k,t)}{dt} z_1^{k_1}z_2^{k_2}z_3^{k_3} = -\left\{\lambda(t) + \sum_{i=1}^{3}\mu_i(t)\right\}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} P(k,t)z_1^{k_1}z_2^{k_2}z_3^{k_3} +$$

$$+\lambda(t)\sum_{i=1}^{2} p_{0i}u(k_i)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} P(k - I_i, t)z_1^{k_1}z_2^{k_2}z_3^{k_3} + \sum_{i=1}^{2}\mu_i(t)u(k_3)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} P(k + I_i - I_3, t)z_1^{k_1}z_2^{k_2}z_3^{k_3} +$$

$$+\mu_3(t)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} P(k + I_3, t)z_1^{k_1}z_2^{k_2}z_3^{k_3}$$

We will examine some sums from the right side. Let

$$\sum\nolimits_2 (z,t) = \sum_{i=1}^{2} p_{0i}u(k_i)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} P(k - I_i, t)z_1^{k_1}z_2^{k_2}z_3^{k_3}$$

Then

$$\sum\nolimits_2 (z,t) = p_{01}z_1\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} P(k_1 - 1, k_2, k_3, t)z_1^{k_1-1}z_2^{k_2}z_3^{k_3} +$$

$$+ p_{02}z_2\sum_{k_1=0}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=0}^{\infty}P(k_1,k_2-1,k_3,t)z_1^{k_1}z_2^{k_2-1}z_3^{k_3} =$$

$$= p_{01}z_1\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}P(k_1,k_2,k_3,t)z_1^{k_1}z_2^{k_2}z_3^{k_3} + p_{02}z_2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}P(k_1,k_2,k_3,t)z_1^{k_1}z_2^{k_2}z_3^{k_3} =$$

$$= (p_{01}z_1 + p_{02}z_2)\Psi_3(z,t) = \sum_{i=1}^{2} p_{0i}z_i\Psi_3(z,t)$$

For the sum $\sum_3(z,t) = \sum_{i=1}^{2}\mu_i(t)u(k_3)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}P(k+I_i-I_3,t)z_1^{k_1}z_2^{k_2}z_3^{k_3}$ we get:

$$\sum_3(z,t) = \mu_1(t)u(k_3)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}P(k_1+1,k_2,k_3-1,t)z_1^{k_1}z_2^{k_2}z_3^{k_3} +$$

$$+\mu_2(t)u(k_3)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}P(k_1,k_2+1,k_3-1,t)z_1^{l}z_2^{m}z_3^{r} =$$

$$=\mu_1(t)\frac{z_3}{z_1}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=1}^{\infty}P(k_1+1,k_2,k_3-1,t)z_1^{k_1+1}z_2^{k_2}z_3^{k_3-1} +$$

$$+\mu_2(t)\frac{z_3}{z_2}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=1}^{\infty}P(k_1,k_2+1,k_3-1,t)z_1^{k_1}z_2^{k_2+1}z_3^{k_3-1} =$$

$$=\mu_1(t)\frac{z_3}{z_1}\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}P(k_1,k_2,k_3,t)z_1^{k_1}z_2^{k_2}z_3^{k_3} + \mu_2(t)\frac{z_3}{z_2}\sum_{k_1=0}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=0}^{\infty}P(k_1,k_2,k_3,t)z_1^{k_1}z_2^{k_2}z_3^{k_3} =$$

$$= z_3\left(\frac{\mu_1(t)}{z_1}+\frac{\mu_2(t)}{z_2}\right)\Psi_3(z,t) - \mu_1(t)\frac{z_3}{z_1}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}P(0,k_2,k_3,t)z_2^{k_2}z_3^{k_3} -$$

$$-\mu_2(t)\frac{z_3}{z_3}\sum_{k_1=0}^{\infty}\sum_{k_3=0}^{\infty}P(k_1,0,k_3,t)z_1^{k_1}z_3^{k_3}$$

And finally for the last sum $\sum_4(z,t) = \mu_3(t)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}P(k_1,k_2,k_3+1,t)z_1^{k_1}z_2^{k_2}z_3^{k_3}$ we can have:

$$\sum_4(z,t) = \frac{\mu_3(t)}{z_3}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}P(k_1,k_2,k_3+1,t)z_1^{k_1}z_2^{k_2}z_3^{k_3+1} =$$

$$= \frac{\mu_3(t)}{z_3} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=1}^{\infty} P(k_1,k_2,k_3,t) z_1^{k_1} z_2^{k_2} z_3^{k_3} = \frac{\mu_3(t)}{z_3} \Psi_3(z,t) - \frac{\mu_3(t)}{z_3} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} P(k_1,k_2,0,t) z_1^{k_1} z_2^{k_2}$$

Thus, taking into account designation (2), we obtain heterogeneous linear differential equation for generating function

$$\frac{d\Psi_3(z,t)}{dt} = -\left[\lambda(t) + \sum_{i=1}^{3} \mu_i(t) - \lambda(t)\left(p_{01}z_1 + p_{02}z_2\right) - z_3\left(\frac{\mu_1(t)}{z_1} + \frac{\mu_2(t)}{z_2}\right) - \frac{\mu_3(t)}{z_3}\right]\Psi_3(z,t) -$$

$$-\mu_1(t)\frac{z_3}{z_1} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} P(0,k_2,k_3,t) z_2^{k_2} z_3^{k_3} - \mu_2(t)\frac{z_3}{z_2} \sum_{k_1=0}^{\infty} \sum_{k_3=0}^{\infty} P(k_1,0,k_3,t) z_1^{k_1} z_3^{k_3} -$$

$$-\frac{\mu_3(t)}{z_3} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} P(k_1,k_2,0,t) z_1^{k_1} z_2^{k_2} \tag{3}$$

Since all QS functioning in heavy load conditions, last three addends in equation (3) are equal to zero and equation becomes homogeneous:

$$\frac{d\Psi_3(z,t)}{dt} = -\left[\lambda(t) + \sum_{i=1}^{3} \mu_i(t) - \lambda(t)\left(p_{01}z_1 + p_{02}z_2\right) - z_3\left(\frac{\mu_1(t)}{z_1} + \frac{\mu_2(t)}{z_2}\right) - \frac{\mu_3(t)}{z_3}\right]\Psi_3(z,t) \tag{4}$$

We will consider that QN was free at initial state, i.e. $P(0,0,0,0) = 1$. Then starting condition for the last equation will be $\Psi_3(z,0) = 1$. So, we get homogeneous equation (4) for the generating function. Solving it with starting condition we obtain:

$$\Psi_3(z,t) = a_0(t) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \frac{\left(\Lambda(t) - \Lambda(0)\right)^{i+j} p_{01}^i p_{02}^j}{i!\, j!\, l!\, s!\, q!} \times$$

$$\times \left(\mathrm{M}_1(t) - \mathrm{M}_1(0)\right)^l \left(\mathrm{M}_2(t) - \mathrm{M}_2(0)\right)^s \left(\mathrm{M}_3(t) - \mathrm{M}_3(0)\right)^q z_1^{i-l} z_2^{j-s} z_3^{l+s-q} \tag{5}$$

where $\Lambda(t) = \int \lambda(t)dt, \ \mathrm{M}_i(t) = \int \mu_i(t)dt, \ i = \overline{1,3},$

$$a_0(t) = \exp\left[-\left(\Lambda(t) - \Lambda(0)\right) - \sum_{i=1}^{3} \left(\mathrm{M}_i(t) - \mathrm{M}_i(0)\right)\right]$$

Example 1. Let's examine case when service rates $\lambda(t), \ \mu_i(t), \ i = \overline{1,n},$ do not depend on time, i.e. $\lambda(t) = \lambda, \ \ \mu_i(t) = \mu_i, \ \ i = \overline{1,n}.$ Then $\Lambda(t) = \lambda t, \ \ \mathrm{M}_i(t) = \mu_i t,$

$i=\overline{1,3}.$ We will find, for example, probability state $P(1,1,1,t),$ which equals to coefficient of $z_1z_2z_3$ in decomposition of $\Psi_3(z,t).$ Thus degrees at $z_1z_2z_3$ should satisfy to relationships:

$$\begin{cases} i-l=1, \\ j-s=1, \\ l+s-q=1, \end{cases} \quad \text{i.e.} \quad \begin{cases} l=i-1, \\ s=j-1, \\ q=i+j-3, \\ i,j>0. \end{cases}$$

So in this case, following from (5),

$$P(1,1,1,t)=e^{-\left(\lambda+\sum_{i=1}^{3}\mu_i\right)t}\sum_{\substack{i,j=0,\\ i+j>2}}^{\infty}\frac{\lambda^{i+j}p_{01}^{j}p_{02}^{j}\mu_1^{i-1}\mu_2^{j-1}\mu_3^{i+j-3}}{i!\,j!(i-1)!(j-1)!(i+j-3)!}t^{3i+3j-5}$$

3. Network state probabilities with arbitrary *n*

Now we will generalize our results for the case of arbitrary number of queueing systems, but, like before, we will suppose that systems are single-line. Then set of equations for state probabilities (1) will look like:

$$\frac{dP(k,t)}{dt}=-\left\{\lambda(t)+\sum_{i=1}^{n}\mu_i(t)\right\}P(k,t)+\lambda(t)\sum_{i=1}^{n-1}p_{0i}u(k_i)P(k-I_i,t)+$$

$$+\sum_{i=1}^{n-1}\mu_i(t)u(k_n)P(k+I_i-I_n,t)+\mu_n(t)P(k+I_n,t)$$

As before, let us assume that network systems functioning in heavy load conditions, i.e. $k_i(t)>0\ \ \forall t>0,\ \ i=\overline{1,n}.$ We will define n-dimensional generating function:

$$\Psi_n(z,t)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k_1,k_2,..,k_n,t)\prod_{i=1}^{n}z_i^{k_i} \qquad (6)$$

Theorem. *Expression for generating function (6) looks like*

$$\Psi_n(z,t)=a_0(t)\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\mathrm{K}\sum_{l_{n-1}=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\mathrm{K}\sum_{r_{n-1}=0}^{\infty}\sum_{q=0}^{\infty}\frac{(\Lambda(t)-\Lambda(0))^{l_1+l_2+\mathrm{K}+l_{n-1}}p_{01}^{l_1}p_{02}^{l_2}\cdot\mathrm{K}\cdot p_{0n-1}^{l_{n-1}}}{l_1!l_2!\cdot\mathrm{K}\cdot l_{n-1}!r_1!r_2!\cdot\mathrm{K}\cdot r_{n-1}!q!}\times$$

$$\times\left(\mathrm{M}_1(t)-\mathrm{M}_1(0)\right)^{r_1}\left(\mathrm{M}_2(t)-\mathrm{M}_2(0)\right)^{r_2}\cdot \mathrm{K}\cdot\left(\mathrm{M}_{n-1}(t)-\mathrm{M}_{n-1}(0)\right)^{r_{n-1}}\left(\mathrm{M}_n(t)-\mathrm{M}_n(0)\right)^{q}\times$$

$$\times z_1^{l_1-r_1}z_2^{l_2-r_2}\cdot \mathrm{K}\cdot z_{n-1}^{l_{n-1}-r_{n-1}}z_n^{r_1+r_2+\mathrm{K}+r_{n-1}-q} \tag{7}$$

where $a_0(t)=\exp\left\{-\left(\Lambda(t)-\Lambda(0)\right)-\sum_{i=1}^{n}\left(\mathrm{M}_i(t)-\mathrm{M}_i(0)\right)\right\}.$

Proof. Doing investigations similar to the case n = 3, we obtain homogeneous linear differential equation for the function (6):

$$\frac{d\Psi_n(z,t)}{dt}=-\left[\lambda(t)+\sum_{i=1}^{n}\mu_i(t)-\lambda(t)\sum_{i=1}^{n-1}p_{0i}z_i-z_n\sum_{i=1}^{n-1}\frac{\mu_i(t)}{z_i}-\frac{\mu_n(t)}{z_n}\right]\Psi_n(z,t)$$

The solution of this equation with starting condition $\Psi_n(z,0)=1$ is function

$$\Psi_n(z,t)=\exp\left\{-\left[\left(\Lambda(t)-\Lambda(0)\right)+\sum_{i=1}^{n}\left(\mathrm{M}_i(t)-\mathrm{M}_i(0)\right)-\left(\Lambda(t)-\Lambda(0)\right)\sum_{i=1}^{n-1}p_{0i}z_i-\right.\right.$$

$$\left.\left.-\sum_{i=1}^{n-1}\left(\mathrm{M}_i(t)-\mathrm{M}_i(0)\right)\frac{z_n}{z_i}-\frac{\mathrm{M}_n(t)-\mathrm{M}_n(0)}{z_n}\right]\right\}$$

Let’s re-arrange (7) to form, convenient for state probabilities obtaining:

$$\Psi_n(z,t)=a_0(t)\exp\left\{\left(\Lambda(t)-\Lambda(0)\right)\sum_{i=1}^{n-1}p_{0i}z_i\right\}\exp\left\{z_n\sum_{i=1}^{n-1}\frac{\mathrm{M}_i(t)-\mathrm{M}_i(0)}{z_i}\right\}\exp\left\{\frac{\mathrm{M}_n(t)-\mathrm{M}_n(0)}{z_n}\right\}=$$

$$=a_0(t)a_1(z,t)a_2(z,t)a_3(z,t)$$

where

$$a_1(z,t)=\exp\left\{\left(\Lambda(t)-\Lambda(0)\right)\sum_{i=1}^{n-1}p_{0i}z_i\right\}=$$

$$=\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\mathrm{K}\sum_{l_{n-1}=0}^{\infty}\frac{\left(\Lambda(t)-\Lambda(0)\right)^{l_1+l_2+\mathrm{K}+l_{n-1}}}{l_1!l_2!\cdot\mathrm{K}\cdot l_{n-1}!}p_{01}^{l_1}p_{02}^{l_2}\cdot\mathrm{K}\cdot p_{0n-1}^{l_{n-1}}z_1^{l_1}z_2^{l_2}\cdot\mathrm{K}\cdot z_{n-1}^{l_{n-1}}$$

$$a_2(z,t)=\exp\left\{z_n\sum_{i=1}^{n-1}\frac{\mathrm{M}_i(t)-\mathrm{M}_i(0)}{z_i}\right\}=$$

$$=\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\mathrm{K}\sum_{r_{n-1}=0}^{\infty}\frac{z_n^{r_1+r_2+\mathrm{K}+r_{n-1}}}{r_1!r_2!\cdot\mathrm{K}\cdot r_{n-1}!}\left(\mathrm{M}_1(t)-\mathrm{M}_1(0)\right)^{r_1}\left(\mathrm{M}_2(t)-\mathrm{M}_2(0)\right)^{r_2}\cdot\mathrm{K}\cdot\left(\mathrm{M}_{n-1}(t)-\mathrm{M}_{n-1}(0)\right)^{r_{n-1}}\times$$

$$\times z_1^{-r_1}z_2^{-r_2}\cdot\mathrm{K}\cdot z_{n-1}^{-r_{n-1}}$$

$$a_3(z,t) = \exp\left\{\frac{\mathrm{M}_n(t) - \mathrm{M}_n(0)}{z_n}\right\} = \sum_{q=0}^{\infty} \frac{\left[\left(\mathrm{M}_n(t) - \mathrm{M}_n(0)\right) z_n^{-1}\right]^q}{q!}$$

Multiplying of these functions gives us required result.

Example 2. Let's obtain, for example, in the case when $\lambda(t) = \lambda$, $\mu_i(t) = \mu_i$, $i = \overline{1, n}$, state probability $P(1,1,1,...,1,t)$, which equal to coefficient of $\prod_{i=1}^{n} z_i$ in decomposition of $\Psi_n(z,t)$. Let's compose relationships for degrees:

$$\begin{cases} l_1 - r_1 = 1, \\ l_2 - r_2 = 1, \\ \quad \ldots\ldots\ldots \\ l_{n-1} - r_{n-1} = 1, \\ r_1 + r_2 + \ldots + r_{n-1} - q = 1, \end{cases} \qquad \text{i.e.} \qquad \begin{cases} r_i = l_i - 1, \\ q = \sum_{i=1}^{n-1} l_i - 1. \end{cases}$$

Then, according (7),

$$P(1,1,\ldots,1,t) = a_0(t) \times$$

$$\times \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \ldots \sum_{l_{n-1}=1}^{\infty} \frac{\lambda^{l_1+l_2+\ldots+l_{n-1}} \cdot p_{01}^{l_1} p_{02}^{l_2} \cdot \ldots \cdot p_{0n-1}^{l_{n-1}} \cdot \mu_1^{l_1-1} \mu_2^{l_2-1} \cdot \ldots \cdot \mu_{n-1}^{l_{n-1}-1} \mu_n^{l_1+l_2+\ldots+l_{n-1}-n}}{l_1! l_2! \cdot \ldots \cdot l_{n-1}! (l_1-1)! (l_2-1)! \cdot \ldots \cdot (l_{n-1}-1)! (l_1+l_2+\ldots+l_{n-1}-n)!} \times$$

$$\times t^{3l_1+3l_2+\ldots+3l_{n-1}-2n+1}$$

References

[1] Matalytski M., Gomanchuk S., Pankow A., Analysis of open queueing networks using method of generating functions, Scientific Research of the Institute of Mathematics and Computer Science, Czestochowa University of Technology 2002, 1(1), 143-152.