ON TOLERANCE AVERAGED MODEL OF THE PROPAGATION OF BOUNDARY DISPLACEMENT FLUCTUATIONS IN FGM-TYPE LAMINATED COMPOSITE

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Abstract. The object of the contribution is the analysis of a certain initial-boundary problems in elastodynamics of FGM-type multiphased laminated composites. The aim of the paper is to formulate an answer to the question what kind of assumptions are enough to be taken into account in the framework of the simplified tolerance model for the elastodynamics of the considered laminates the boundary effect will be observed.

Introduction

The aim of the contribution is to discuss of the tolerance description of behaviours observed in the linear elastodynamics of multiphased FGM-type laminates being a certain effect of displacement perturbations situated on the boundary of the region occupied by the laminated composite material. To this end the second order tolerance model which is also named a simplified tolerance model is taken into account [1, 2]. It is a certain adaptation of the classical tolerance averaged model to the analysis of problems like boundary effect problems in elastodynamics of a laminated medium.

In the framework of the contribution the latin indices i, j, k, l, p, q, r, s run over 1, 2, 3, greek indices $\alpha, \beta, \gamma,...$ r over 1, 2, latin indices $A,B,C,...$ over 1, ..., N, where N a number of shape function which should be postulated in every problem analysed in the framework of the tolerance averaging technique approach.

1. Model equations

The starting point of the considerations is the tolerance averaging model equation system of the FGM-type laminated composite. It will be assumed that the Carthesian coordinate system have been introduced and the physical 3-dimensional space and Ox_1-axis direction is perpendicular to the laminae interfaces planes. The microstructure of the FGM-type laminated composite determined by the length λ

of the repeated layer and thickness $l^A(z)$ of the A-th laminae, $A = 1,...,N$. We shall introduce averaging operator

$$\langle f \rangle(z) = \frac{1}{l}\int_{z-l/2}^{z+l/2} f(y)dy, \quad z \in (-L+l/2, L-l/2)$$

where $f(\cdot)$ is an integrable function defined in the interval (z-$l/2$, z+$l/2$). The basic unknowns of this system is the averaged displacement field $\mathbf{u}$ together with the fluctuation amplitudes $\mathbf{v}^A$, $A = 1,...,N$. The displacement field $\mathbf{U}$ of the laminated composite can be determined under the formula

$$\mathbf{U}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) + h^A(x_1)\mathbf{v}^A(\mathbf{x},t) \tag{1}$$

in which $\mathbf{x} = (x_1, \overline{\mathbf{x}})$, $\overline{\mathbf{x}} = (x_1, x_2)$, and $h^A = lg^A$, $A = 1,...,N$, are locally-periodic functions named shape functions which form and a number are strictly connected with the microstructure of the FGM-type laminated composite. For particulars the reader is referred to [3]. Denoting by $\rho(\cdot)$ and $C_{ijkl}(\cdot)$ the mass density field and elasticity modulus tensor fields, respectively, the tolerance model equations of the elastodynamics of FGM-type laminated composite can be written in the form

$$\begin{aligned}
&\langle\rho\rangle\ddot{u}_i - (\langle C_{ijkl}\rangle u_{l},_{k} + \langle C_{ij1l}\, g^A,_1\rangle \upsilon^A{}_l),_j - l\langle C_{ij\gamma l}\, g^A\rangle \upsilon^A{}_l,_\gamma = 0 \\
&\underline{l^2\langle\rho g^A g^B\rangle\ddot{\upsilon}^A_i} - \underline{l^2\langle C_{ij\gamma l}\, g^A g^B\rangle \upsilon^B{}_l,_{\gamma j}} + \underline{l(\langle g^A,_1 C_{i1\gamma l}\, g^B\rangle - \langle g^A C_{i\gamma 1l}\, g^B,_1\rangle} + \\
&\underline{-l\langle g^A C_{i\beta\gamma l}\, g^B\rangle,_\beta})\upsilon^B{}_l,_\gamma + (\langle g^A,_1 C_{i11l}\, g^B,_1\rangle - \langle g^A C_{i\beta 1l}\, g^B,_1\rangle,_\beta)\upsilon^B{}_l + \\
&+\langle C_{i1kl}\, g^A,_1\rangle u_l,_k - l(< g^A C_{ijkl} > u_l,_k),_j = 0
\end{aligned} \tag{2}$$

Considerations will be restricted to the case in which material gradation of the laminate is situated along the Ox_1-axis direction. It is mean that coefficients in (2) depend on variable x_1 as on a certain parameter. Hence, model equations system (2) can be reformulated to the form

$$\begin{aligned}
&\langle\rho\rangle\ddot{u}_i - (\langle C_{ijkl}\rangle u_l,_k + [C]^A_{ijl}\,\upsilon^A{}_l),_j - l\langle C_{ij\gamma l}\, g^A\rangle \upsilon^A{}_l,_\gamma = 0 \\
&\underline{l^2 r^{AB}\ddot{\upsilon}^A_i} - \underline{l^2 c^{AB}_{i\beta\gamma l}\,\upsilon^B{}_l,_{\gamma\beta}} + \underline{l s^{AB}_{i\beta l}\,\upsilon^B{}_l,_\beta} + \{C\}^{AB}_{il}\,\upsilon^B{}_l + [C]^A_{ikl}\, u_l,_k - \\
&-l < g^A C_{i\beta kl} > u_l,_{k\beta} = 0
\end{aligned} \tag{3}$$

in which coefficients are given by

$$\begin{aligned}
&[C]^A_{ijl} = \langle C_{ij1l}\, g^A,_1\rangle, \quad s^{AB}_{i\beta l} = \langle C_{i1\beta l}\, g^A,_1 g^B - C_{i\beta 1l}\, g^A g^B,_1\rangle \\
&c^{AB}_{ijkl} = \langle C_{ijkl}\, g^A g^B\rangle, \quad \{C\}^{AB}_{il} = \langle g^A,_1 C_{i11l}\, g^B,_1\rangle, \quad r^{AB} = \langle\rho g^A g^B\rangle
\end{aligned} \tag{4}$$

It must be emphasized that

$$\{C\}_{li}^{BA} = \{C\}_{il}^{AB} \tag{5}$$

Now we shall to pass to the simplified model. To this end fluctuation amplitude υ_i^A will be replaced by the alternative variable w_i^A named as internal fluctuation amplitude and interrelated with υ_i^A by the formula

$$\upsilon_i^A = \upsilon^{\hom A}{}_i + w_i^A \tag{6}$$

in which gradient part of the fluctuation amplitude is determined by

$$\langle (g^A{}_{,1})^2 C_{i11j} \rangle \upsilon^{\hom A}{}_j + \langle g^A{}_{,1} C_{i1kl} \rangle u_{k,l} = 0$$

Hence, second equations from (3) takes the form

$$l^2 r^{AB} \ddot{w}_i^B - l^2 (c_{i\beta\gamma l}^{AB} w^B{}_{l,\gamma})_{,\beta} + l s_{i\beta l}^{AB} w^B{}_{l,\beta} + \{C\}_{il}^{AB} w^B{}_l = -l^2 res^{*A}{}_i \tag{7}$$

where

$$res^{*A}{}_i \in O(l) \tag{8}$$

Since $res^{*A}{}_i \in O(l)$ the right hand side of (7) is of an order O(l^3) and hence can be omitted. Denoting by K_{pr}^{AD} the inverse matrix to $\{C\}_{rq}^{DB}$, i.e.

$$K_{pr}^{AD} \{C\}_{rq}^{DB} = \delta^{AB} \delta_{pq}$$

and introducing denotation

$$C^{eff}{}_{ijkl} = \langle C_{ijkl} \rangle - [C]_{ijp}^A K_{pq}^{AB} [C]_{qkl}^A \tag{9}$$

model equations system can be approximately rewritten as

$$\begin{aligned} &\langle \rho \rangle \ddot{u}_i - (C_{ijkl}^{eff} u_{k,l})_{,j} = [C]_{ikl}^A w^A{}_{k,l} \\ &l^2 [r^{AB} \ddot{w}_i^A - c_{ij\beta\gamma l}^{AB} w_{l,\gamma\beta}^B] + l s_{i\beta l}^{AB} w^B{}_{l,\beta} + \{C\}_{il}^{AB} w_l^B = 0 \end{aligned} \tag{10}$$

At the same time formula (1) for total displacements U_i takes the form

$$U_i = u_i - l g^A K_{ip}^{AB} \langle g^A{}_{,1} C_{p1rs} \rangle u_{s,r} + l g^A w^A{}_i + O(\lambda^3) \tag{11}$$

It must be emphasized that in the special case of two-phased periodic linear elastic laminated composite with exclusively shape function $h = h^1$ takes the form

$$\begin{aligned}&\langle\rho\rangle\ddot{u}_i - (C^{eff}{}_{ijkl}u_k,_l),_j = [C]_{ikl}w\,{}_k,_l\\&l^2[r\ddot{w}_i - c_{ij\beta\gamma l}w\,{}_l,_{\gamma\beta}] + ls_{i\beta l}w_l,_\beta\{C\}_{il}w_l = 0\end{aligned}\tag{12}$$

where

$$\begin{aligned}&w_i = w^1{}_i, \quad [C]_{ikl} = [C]^1_{ikl},\\&r = r^{11}, \quad c_{ij\beta\gamma l} = c^{11}_{ij\beta\gamma l}, \quad \{C\}_{il} = \{C\}^{11}_{il}\end{aligned}$$

and $s_{i\beta l} = s^{11}_{i\beta l} = 0$. Moreover, (11) reduces to the form

$$U_i = u_i - lgK_{ip}\langle g,_1 C_{p1rs}\rangle u_s,_r + lgw_i + O(\lambda^3)$$

for $g = g^1$, $K_{ip} = K^{11}_{ip}$. Equations (10) represent simplified tolerance model of the linear elastodynamics of FGM-type laminated medium.

It is the well known fact that that term $ls_{i\beta l}w_l,_\beta$ in the second of the above equation in the case of the saw-like function vanish [4], but in general case this property cannot be satisfied. For example for the shape function defined as the periodic solution related to the lower frequency ω to the problem

$$\begin{cases}\omega^2\rho(x_1)\ddot{h}(x_1) + C_{1111}(x_1)h(x_1) = 0 & for \quad x_1 \in (0,L)\setminus\{nl: \quad n = 0,1,...,L/l\}\\ [C_{1111}h,_1](x_1) = 0 & for \quad x_1 = nl, \quad n = 0,1,...,L/l\end{cases}\tag{13}$$

we have $s_{i\beta l} \neq 0$ for any admissible indices α, β, l.

2. Formulation of the problem

Let us consider the FGM-type laminated linear-elastic composite which occupy in the reference configuration the region Ω = $(-L,L)\times\Pi$, where Π is a certain region in R^2. For Π = $(0,H)\times(0,H_1)$ laminated composite under consideration is illustrated on Figure 1.

The aim of the paper is the analysis of the following initial-boundary problem for internal fluctuation amplitudes w_i^A

(P_w) *Find amplitudes* $\alpha^A{}_i = \alpha^A{}_i(x_1,\mathbf{x})$, $(x_1,\mathbf{x})\in\bar{\Omega}$, *such that vibrations* $w_i^A(\cdot,t) = \alpha_i^A(\cdot)\cos\omega t$ *of internal fluctuation amplitude* $w_i^A(\cdot,t)$ *satisfy in* Ω *the second from equations* (10) *and are a certain expansion into the region* Ω *of the boundary perturbations*

$$w^A{}_i = \alpha^A{}_i(x_1,\mathbf{x})\cos\omega t, \quad (x_1,\mathbf{x})\in\partial\Omega, \quad t>0$$

of internal fluctuation amplitude $w_i^A(\cdot,t)$.

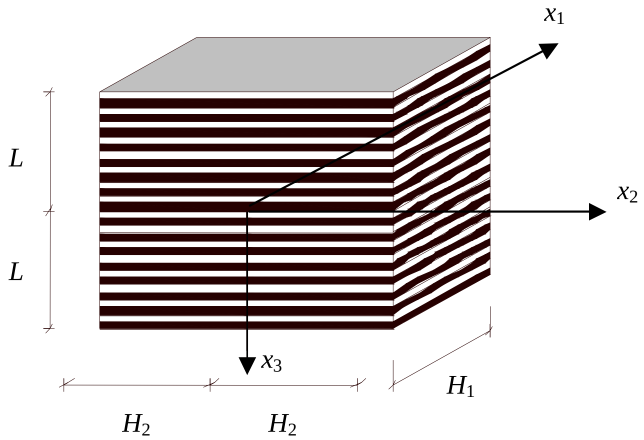

Fig. 1. Example of a laminated solid

In the special case mentioned at the end of Sec. 1 in which we deal with exclusively one saw-like shape function for two-phased linear-elastic the periodic laminated composite, the above problem was analyzed in [1, 2]. For two-phased FGM-laminated composite case the problem was discussed in [3].

The aim of this paper is to discuss this problem from mathematical viewpoint for multi-phased FGM-type laminated linear elastic composites. To this end an alternative form of the simplified model which will be more useful for the analysis will be obtained. It can be obtained by exponential transformation given by the formula

$$w^A{}_k = (e^{Qx_\alpha})^{AB}_{ij} \pi^B{}_j$$

where

$$e^Q \equiv \sum_{n=0}^{\infty} \frac{Q^n}{n!}, \quad (Q^n)^{AB}_{ij} = Q^{AC_1}_{ip_1} Q^{C_1C_2}_{p_1p_2} \ldots Q^{C_{n-2}C_{n-1}}_{p_{n-2}p_{n-1}} Q^{C_{n-1}B}_{p_{n-1}j},$$

$$p_1, \ldots, p_{n-1} = 1, 2, 3, \quad C_1, \ldots, C_{n-1} = 1, \ldots, N$$

for

$$Q^{AB}_{ij} = s^{AB}_{ij\alpha} x_\alpha$$

under which equations (10) yield

$$\begin{aligned} &\langle \rho \rangle \ddot{u}_i - (C^{eff}{}_{ijkl} u_k{,}_l){,}_j = [C]^A_{ikl} (e^{Qx_\alpha})^{AB}_{kp} \pi^B{}_p{,}_l \\ &l^2 [r^{AB} \ddot{\pi}^B_i - c^{AB}_{i\beta\gamma l} \pi^B{}_l{,}_{\gamma\beta}] + M^{AB}_{il} \pi^B{}_l = 0 \end{aligned} \tag{14}$$

where

$$M_{ij}^{AB} = \{C\}_{ij}^{AB} + 0.25 s_{i\alpha p}^{AC} (c^{-1})_{p\alpha\beta s}^{CD} s_{j\beta s}^{DB}, \quad (c^{-1})_{p\alpha\beta s}^{CD} c_{s\beta\gamma q}^{DE} = \delta^{CE} \delta_{pq} \delta_{\alpha\gamma} \tag{15}$$

The basic unknowns of this new form of simplified tolerance system are averaged displacements u_i and introducing above knew unknown $\pi^B{}_j$ which will be called *generalized internal fluctuation amplitude*. Now, we can reformulate problem (P_w) to the form

(P_π) *Find amplitudes* $\beta^A{}_i = \beta^A{}_i(x_1, \mathbf{x}), \quad (x_1, \mathbf{x}) \in \bar{\Omega}$, *such that vibrations* $\pi_i^A(\cdot,t) = \beta_i^A(\cdot)\cos\omega t$ *of generalized internal fluctuation amplitude* $\pi_i^A(\cdot,t)$ *satisfy in* Ω *the second from equations (13) and are a certain expansion into the region* Ω *of the boundary perturbations*

$$\pi^A{}_i = \beta^A{}_i(x_1, \mathbf{x})\cos\omega t, \quad (x_1, \mathbf{x}) \in \partial\Omega, \quad t > 0$$

of internal fluctuation amplitude $\pi_i^A(\cdot,t)$.

It is easy to verify that if $\beta^A{}_i = \beta^A{}_i(x_1, \mathbf{x}), \quad (x_1, \mathbf{x}) \in \bar{\Omega}$, is a certain solution to the problem (P_π) then $\alpha^A{}_k = (e^{Qx_\alpha})_{ij}^{AB} \beta^B{}_j$ is a certain solution to the problem (P_w) and vice versa and hence we can restrict considerations to the analysis of the problem (P_π).

3. Special case

In the framework of this paper we are to discuss this problem in the special case of two-phased laminated composites. Under the above assumption we shall apply exclusively one shape function $h = h^1$ and under denotations $\pi = \pi^1$, $[C]_{ikl} = [C]_{ikl}^1$, $r = r^{11}$, $c_{ij\beta\gamma l} = c_{ij\beta\gamma l}^{11}$, $M_{il} = M_{il}^{11}$, $(e^{Qx_\alpha})_{kp} = (e^{Qx_\alpha})_{kp}^{11}$, equations (14) are reduced to

$$\begin{aligned} &\langle\rho\rangle \ddot{u}_i - (C^{eff}{}_{ijkl} u_{k},_{l}),_{j} = [C]_{ikl} (e^{Qx_\alpha})_{kp} \pi_{p},_{l} \\ &\lambda^2 (r\ddot{\pi}_i - c_{i\beta\gamma l} \pi_{l},_{\gamma\beta}) + M_{il} \pi_l = 0 \end{aligned} \tag{16}$$

for

$$\begin{aligned} &M_{ij} = M_{ij}^{11} = \{C\}_{ij} + 0.25 s_{i\alpha p} (c^{-1})_{p\alpha\beta s} s_{j\beta s}, \quad \{C\}_{ij} = \{C\}_{ij}^{11}, \\ &s_{j\beta s} = s_{j\beta s}^{11}, \quad (c^{-1})_{p\alpha\beta s} = (c^{-1})_{p\alpha\beta s}^{11} \end{aligned} \tag{17}$$

Moreover M_{ij} is a certain symmetric matrix and hence problem (P_π) is equivalent to the eigenvalue problem

$$\lambda^2 c_{ij\beta\gamma}\pi_{l},_{\gamma\beta} + (\lambda^2 r\omega^2 - M_{il})\pi_l = 0 \tag{18}$$

4. Illustrative example

To illustrate considerations we shall assume that the planes $x_\alpha = const.$ are symmetry planes in every constituent of the considered laminated composite. In this case we shall observe that similarly as in the case of saw-like function the equation (18) separate into two independent scalar equations for π_1 and π_2

$$\begin{aligned} l^2 c_{1221}\pi_1,_{22} + (l^2 r\omega^2 - M_{11})\pi_1 = 0 \\ l^2 c_{2222}\pi_2,_{22} + (l^2 r\omega^2 - M_{22})\pi_2 = 0 \end{aligned} \tag{19}$$

every of which posses the same mathematical form. Hence we can rewrite $(19)_1$ and $(19)_2$ in the nondimensional form

$$\lambda^2 \frac{d^2\zeta}{d\xi^2} + \gamma^2(1-\Omega^2)\zeta = 0 \tag{20}$$

with respect to a certain length entity d, nondimensional variables $\xi = x_2/d,\ \lambda = l/d$ where nondimensional unknown $\zeta(\xi) \in \{\pi_1(\xi d), \pi_2(\xi d)\}$, $0 \le \xi \le H/d$, and denotations

$$\gamma \in \left\{\frac{M_{11}}{c_{1212}}, \frac{M_{22}}{c_{2222}}\right\}, \quad \Omega^2 \in \left\{\frac{\omega^2\lambda^2 r}{M_{11}}, \frac{\omega^2\lambda^2 r}{M_{22}}\right\} \tag{21}$$

Conclusions

In the paper the problem (P_w) of the propagation of the boundary displacement fluctuations into the interior of the region occupied by the multiphased FGM-type laminated composite materials has been analyzed. It was shown that in the special case of the two-phased laminated composite the problem (P_w) can be reformulated to a certain problem (P_π) and separated on this way to the two independent one-dimensional eigenvalue problems (20). Problems of this type was discussed and solved for the periodic case in [2]. For saw-like function taken as the shape function these solutions has been adapted for a certain FGM-type laminates in [3]. An extended version of this paper will be published elsewhere.

References

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