THE POLYNOMIAL TENSOR INTERPOLATION

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Abstract. In this paper the tensor interpolation by polynomials of several variables is considered.

Introduction

The formulas of tensor interpolation by polynomials of several variables are the unknow in the interpolation methods ([1]). Using the Kronecker tensor product of matrices ([2, 3]) the polynomial tensor interpolation formula was given in tis article.

1. The Kronecker product of matrices

The Kronecker product of two matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ of degrees respectivly *m* and *n* we define as a matrix given in block form as:

$$A \otimes B = [a_{ij}B] = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mm}B \end{bmatrix}$$

we denote the Kronecker product of A and B by $A \otimes B$. The Kronecker product is also known as the *direct product* or the *tensor product*.

Some properties for Kronecker products of two matrices:

1. The matrices $A \otimes B$ and $B \otimes A$ are orthogonaly similar, which means that square matrix U exists and $B \otimes A = U^t(A \otimes B)U$, $U^tU = I$.
2. If A,B,C,D are square matrices such that the products AC and BD exist, then $(A \otimes B)(C \otimes D)$ exist and

   $$(A \otimes B)(C \otimes D) = AC \otimes BD$$

   (the "Mixed Product Rule")

3. $(\alpha A) \otimes (\beta B) = \alpha\beta(A \otimes B)$
4. $(A \otimes B)^t = A^t \otimes B^t$
5. If A and B are invertible matrices, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
6. $\det(A \otimes B) = (\det A)^n (\det B)^m$
7. the $(a \otimes b)_{rs}$ element of the matrix $A \otimes B$ is given by the product

$$(a \otimes b)_{rs} = a_{ij} b_{kl}$$

where $r = (i-1)n + k$, $s = (j-1)n + l$.

Next, we consider the quadratic matrices $A_1 = \left[(a_1)_{i_1 j_1}\right], \ldots, A_k = \left[(a_k)_{i_k j_k}\right]$ of degrees respectivly $n_1,..,n_k$ and define the tensor product inductively:

$$A_1 \otimes A_2 \otimes \ldots \otimes A_k = (A_1 \otimes A_2 \otimes \ldots \otimes A_{k-1}) \otimes A_k$$

for $k \geq 3$.

Some properties for Kronecker products:

1. The matrices $A_{\sigma(1)} \otimes A_{\sigma(2)} \otimes \ldots \otimes A_{\sigma(k)}$ and $A_1 \otimes A_2 \otimes \ldots \otimes A_k$, where σ is any determine permutation of numbers $1,\ldots,k$, are orthogonaly similar. It means that square matrix U_σ exists and $A_{\sigma(1)} \otimes A_{\sigma(2)} \otimes \ldots \otimes A_{\sigma(k)} =$ $= U_\sigma^t (A_1 \otimes A_2 \otimes \ldots \otimes A_k) U_\sigma$, ${U_\sigma}^t U_\sigma = I$.
2. $(A_1 \otimes A_2 \otimes \ldots \otimes A_k)(B_1 \otimes B_2 \otimes \ldots \otimes B_k) = (A_1 B_1) \otimes (A_2 B_2) \otimes \ldots \otimes (A_k B_k)$ provided the dimensions of the matrices are such that the various expressions exist (the "Mixed Product Rule").
3. $(\alpha_1 A_1) \otimes (\alpha_2 A_2) \otimes \ldots \otimes (\alpha_k A_k) = \alpha_1 \alpha_2 \ldots \alpha_k (A_1 \otimes A_2 \otimes \ldots \otimes A_k)$
4. $(A_1 \otimes A_2 \otimes \ldots \otimes A_k)^t = A_1^t \otimes A_2^t \otimes \ldots \otimes A_k^t$
5. if $A_1,\ldots,A_k$ are invertible matrices, then:
   $(A_1 \otimes A_2 \otimes \ldots \otimes A_k)^{-1} = A_1^{-1} \otimes A_2^{-1} \otimes \ldots \otimes A_k^{-1}$
6. $\det(A_1 \otimes A_2 \otimes \ldots \otimes A_k) = (\det A_1)^{\hat{n}_1 n_2 \ldots n_k} (\det A_2)^{n_1 \hat{n}_2 \ldots n_k} \ldots (\det A_k)^{n_1 n_2 \ldots \hat{n}_k}$
   where $\hat{n}_j$ is omission.
7. the $(a_1 \otimes a_2 \otimes \ldots \otimes a_k)_{ij}$ element of the matrix $A_1 \otimes A_2 \otimes \ldots \otimes A_k$ is given by the product $(a_1 \otimes a_2 \otimes \ldots \otimes a_k)_{ij} = (a_1)_{i_1 j_1} (a_2)_{i_2 j_2} \ldots (a_k)_{i_k j_k}$, where:
   $i = (i_1 - 1)\hat{n}_1 n_2 \ldots n_k + (i_2 - 1)\hat{n}_1 \hat{n}_2 \ldots n_k + \ldots + (i_{k-1} - 1) n_k + i_k$

2. One of the property for Vandermonde's matrix

Consider the Vandermonde's matrix:

$$V_{p+1}=V_{p+1}\left(X_0,X_1,X_2,...,X_p\right)=\begin{bmatrix}1 & X_0 & \dots & X_0^p\\ 1 & X_1 & \dots & X_1^p\\ \vdots & \vdots & \ddots & \vdots\\ 1 & X_p & \dots & X_p^p\end{bmatrix}$$

of degree $p+1$.

The algebraic complement of the matrix V_{p+1}, has the form

$$D_{ij}=(-1)^{i+j}\tau_{p-j}\left(X_0,...,\hat{X}_i,...,X_p\right)\det V_p\left(X_0,...,\hat{X}_i,...,X_p\right)$$

where $\tau_{p-j}\left(X_0,...,\hat{X}_i,...,X_p\right)$ design the fundamental symmetric τ_{p-j} polynomial of the rank $p-j$ of variables $X_0,...,\hat{X}_i,...,X_p$, and the symbol $\hat{X}_i$ means omitting the variable X_i ($\tau_0=0$). Similarly for Vandermonde's determinant $det V_p$.

This property we easly obtain on the one hand by evolving determinant

$$\det V_{p+1}=\det\begin{bmatrix}1 & X_0 & \dots & X_0^p\\ 1 & X_1 & \dots & X_1^p\\ \vdots & \vdots & \ddots & \vdots\\ 1 & X_p & \dots & X_p^p\end{bmatrix}=\prod\nolimits_{0\le k<l\le p}\left(X_l-X_k\right)$$

according to *i*-th row, and on the other hand by sorting its value according to X_i variable power.

In particular

$$\frac{D_{ij}}{\det V_{p+1}}=\frac{(-1)^{i+j}\tau_{p-j}\left(X_0,...,\hat{X}_i,...,X_p\right)}{\Pi_i}$$

where

$$\Pi_i=\prod\nolimits_{\substack{0\le k<l\le p\\ k\ne i \text{ or } l\ne i}}\left(X_l-X_k\right)=\left(X_p-X_i\right)...\left(X_{i+1}-X_i\right)\left(X_i-X_{i-1}\right)...\left(X_i-X_0\right)$$

3. The polynomial tensor interpolation

The coefficients matrix $\left[a_{j_1\cdots j_k}\right]$ of the polynomial interpolation

$$W\left(X_1,...,X_k\right)=\sum\nolimits_{0\le j_1\le p_1,...,0\le j_k\le p_k}a_{j_1\cdots j_k}X_1^{j_1}...X_k^{j_k}$$

are unknow. The results matrix $\left[(w_1)_{i_1}...(w_k)_{i_k}\right]=\left[w_{i_1...i_k}\right]$ and the nodes matrix $\left[(X_1)_{i_1}\times...\times(X_k)_{i_k}\right]=\left[X_{1i_1}...X_{ki_k}\right]$ are know. The coefficients of the polynomial are determined from the linear system

$$\left(\left[X_{i_1}^{j_1}\right]\otimes...\otimes\left[X_{i_k}^{j_k}\right]\right)vec\left[a_{j_1...j_k}\right]=vec\left[w_{i_1...i_k}\right]$$

where the nodes matrix is the Kronecker product of Vandermonde's matrices $V_{p_1+1}=\left[X_{1i_1}^{j_1}\right],...,V_{p_k+1}=\left[X_{ki_k}^{j_k}\right]$, and the operator "vec" put coefficient matrix $\left[a_{j_1...j_k}\right]$ and results matrix $\left[w_{i_1...i_k}\right]$ in columns, attributing multiindexes $j_1...j_k$ and $i_1...i_k$ properly positions

$$\begin{aligned} j = {} & j_1(\hat{p}_1+1)(p_2+1)...(p_k+1)+j_2(\hat{p}_1+1)(\hat{p}_2+1)...(p_k+1)+ \\ & ...+j_{k-1}(p_k+1)+j_k+1 \end{aligned}$$

and

$$\begin{aligned} i = {} & i_1(\hat{p}_1+1)(p_2+1)...(p_k+1)+i_2(\hat{p}_1+1)(\hat{p}_2+1)...(p_k+1)+ \\ & ...+i_{k-1}(p_k+1)+i_k+1 \end{aligned}$$

of the $a_{j_1...j_k}$ element in coefficients column and $w_{i_1...i_k}$ element in results column. It means that we select ordering

$$\begin{aligned} & 00...00, 00...01, ..., 00...0p_k, 0...1p_k, ..., 00...p_{k-1}p_k, \\ & ......, 0p_2...p_{k-1}p_k, ..., p_1...p_{k-1}p_k \end{aligned}$$

with sequence shown above.
Then the searching coefficients column has a form

$$vec\left[a_{j_1...j_k}\right]=\frac{1}{\det\left[X_{1i_1}^{j_1}\right]}\cdots\frac{1}{\det\left[X_{ki_k}^{j_k}\right]}\left(D_{V_1}\otimes...\otimes D_{V_k}\right)^t vec\left[w_{i_1...i_k}\right]$$

where: $V_1=V_{p_1+1}=\left[X_{1i_1}^{j_1}\right],...,V_k=V_{p_k+1}=\left[X_{ki_k}^{j_k}\right]$.
According to property 7' we obtain the formula for coefficients

$$a_{j_1...j_k}=\sum\nolimits_{0\le i_1\le p_1,...,0\le i_k\le p_k} w_{i_1...i_k}\frac{\left(D_{V_1}\right)_{i_1j_1}}{\det V_1}\cdots\frac{\left(D_{V_k}\right)_{i_kj_k}}{\det V_k}$$

and because fractions shown above has got known form then:

$$a_{j_1\ldots j_k} = \sum\nolimits_{0\le i_1\le p_1,\ldots,0\le i_k\le p_k} (-1)^{I^+ + J^+} w_{i_1\ldots i_k} \frac{\tau_{p_1-j_1}\left(X_{10},\ldots,\hat{X}_{1i_1},\ldots,X_{1p_1}\right)}{\Pi_{1i_1}}$$
$$\ldots\frac{\tau_{p_k-j_k}\left(X_{k0},\ldots,\hat{X}_{ki_k},\ldots,X_{kp_k}\right)}{\Pi_{ki_k}}$$

where: $I^+ = i_1 + \ldots + i_k$, $J^+ = j_1 + \ldots + j_k$ and

$$\Pi_{1i_1} = \left(X_{1p_1} - X_{1i_1}\right)\ldots\left(X_{1i_1+1} - X_{1i_1}\right)\left(X_{1i_1} - X_{1i_1-1}\right)\ldots\left(X_{1i_1} - X_{10}\right)$$

$$\Pi_{ki_k} = \left(X_{kp_k} - X_{ki_k}\right)\ldots\left(X_{ki_k+1} - X_{ki_k}\right)\left(X_{ki_k} - X_{ki_k-1}\right)\ldots\left(X_{ki_k} - X_{k0}\right)$$

And now the polynomial coefficient we can obtain numerically.

References

[1] Kincaid D., Chnej W., Numerical Analysis, Mathematics of Scientific Computing, The University of Texas at Austin, 2002.

[2] Gantmacher F.R., The theory of matrices, Vol. 1, 2, Chelsea 1974.

[3] Lancaster P., Theory of Matries, Acad. Press 1969.

[4] Biernat G., Ciekot A., The polynomial interpolation for technical experiments, Scientific Research of the Institute of Mathematics and Computer Science, Czestochowa University of Technology 2007, 1(6), 19-22.