APPLICATION OF THE RATIONAL B-SPLINE CURVES FOR DESCRIPTION OF COMPLEX DOMAIN SHAPE

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Abstract. In the paper the possibilities of the Non-Uniform Rational B-Splines (NURBS) application for description of domains with complex shape are shown. The B-spline functions, B-spline curves and NURBS curves are taken into account. In the final part of the paper the examples of shape generation using NURBS curves are shown.

Introduction

The curves and surfaces in geometric modelling can be represented by implicit equations and parametric functions. The implicit equation of a curve lying in the plane xy has the form $f(x, y) = 0$. For example, the equation $f(x, y) = x^2 + y^2 - 1 = 0$ describes the circle of unit radius centered at the origin. In parametric form, each of the coordinates of a curve point is represented separately as an explicit function of an independent parameter: $\mathbf{C}(t) = (x(t), y(t))$, $a \leq t \leq b$, where the interval $[a, b]$ is arbitrary one. For example, the circle of unit radius centered at the origin can be written in the form: $\mathbf{C}(t) = (\cos(t), \sin(t))$, $0 \leq t \leq 2\pi$. It should be pointed out that implicit representation is unique up to a constant, while the parametric representation of a curve is not unique.

The parametric form is more natural for designing and representing shape in numerical implementation. Additionally, the curve $\mathbf{C}(t) = (x(t), y(t))$ can be treated as a path traced out by a particle as a function of time, t is the time variable and $[a, b]$ is the time interval. The first and the second derivatives of $\mathbf{C}(t)$ correspond to the velocity and acceleration of the particle.

Among the parametric forms of curves, the most popular are the Bezier, rational Bezier, B-spline and NURBS curves. Non-Uniform Rational B-Splines (NURBS) are widely used for the representation, design and data exchange of geometric information processed by computers.

1. B-spline functions

Let $T = \{t_0, t_1, t_2, \ldots, t_m\}$ be a nondecreasing sequence of real numbers, this means $t_0 \leq t_1 \leq \ldots \leq t_i \leq t_{i+1} \ldots \leq t_m$.

The *i-th* B-spline basic function of *n*-degree is defined as

$$N_{i,0}(t) = \begin{cases} 1, & t_i \leq t \leq t_{i+1} \\ 0, & \text{otherwise} \end{cases} \qquad (1)$$

$$N_{i,n}(t) = \frac{t - t_i}{t_{i+n} - t_i} N_{i,n-1}(t) + \frac{t_{i+n+1} - t}{t_{i+n+1} - t_{i+1}} N_{i+1,n-1}(t)$$

From this definition results that for $n = 1$ one obtains

$$N_{i,1}(t) = \frac{t - t_i}{t_{i+1} - t_i} N_{i,0}(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} N_{i+1,0}(t) \qquad (2)$$

Because

$$N_{i,0}(t) = \begin{cases} 1, & t_i \leq t \leq t_{i+1} \\ 0, & \text{otherwise } t \end{cases} \qquad N_{i+1,0}(t) = \begin{cases} 1, & t_{i+1} \leq t \leq t_{i+2} \\ 0, & \text{otherwise } t \end{cases} \qquad (3)$$

so (Fig. 1)

$$N_{i,1}(t) = \begin{cases} \dfrac{t - t_i}{t_{i+1} - t_i}, & t_i \leq t < t_{i+1} \\ \dfrac{t_{i+2} - t}{t_{i+2} - t_{i+1}}, & t_{i+1} \leq t < t_{i+2} \end{cases} \qquad (4)$$
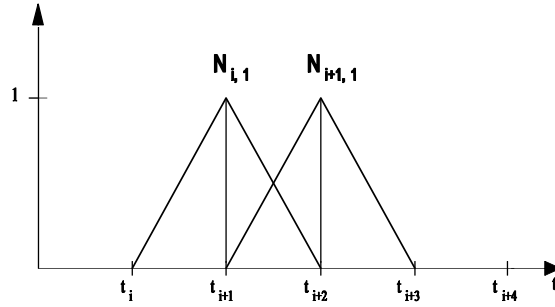

Fig. 1. Functions $N_{i,1}, N_{i+1,1}$

For $n = 2$ one has

$$N_{i,2}(t) = \frac{t - t_i}{t_{i+2} - t_i} N_{i,1}(t) + \frac{t_{i+3} - t}{t_{i+3} - t_{i+1}} N_{i+1,1}(t) \qquad (5)$$

this means (Fig. 2)

$$N_{i,1}(t) = \begin{cases} \dfrac{(t - t_i)^2}{(t_{i+2} - t_i)(t_{i+1} - t_i)} & , & t_i \le t < t_{i+1} \\[3ex] \dfrac{(t - t_i)(t_{i+2} - t)}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} + \dfrac{(t_{i+3} - t)(t - t_{i+1})}{(t_{i+3} - t_{i+1})(t_{i+2} - t_{i+1})} & , & t_{i+1} \le t < t_{i+2} \\[3ex] \dfrac{(t_{i+3} - t)^2}{(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})} & , & t_{i+2} \le t < t_{i+3} \end{cases} \qquad (6)$$
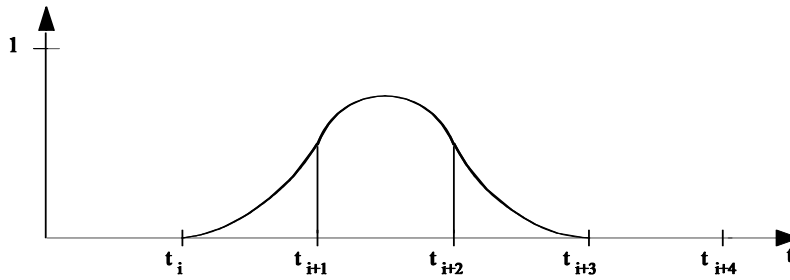

Fig. 2. Function $N_{i,2}$

    In similar way the B-spline basis function is constructed for $n = 3$, 4 etc. It should be pointed out that from the practical point of view the most important is the B-spline basis function for $n = 3$.

    We assume that $n = 2$ and $T = \{0, 0, 0, 1, 1, 1\}$, this means $t_0 = t_1 = t_2 = 0$, $t_3 = t_4 = t_5 = 1$. It is easy to check that in this case only in the interval $[0, 1]$ the nonzero B-spline basis functions exist (c.f. Equation (6), Figure 3)

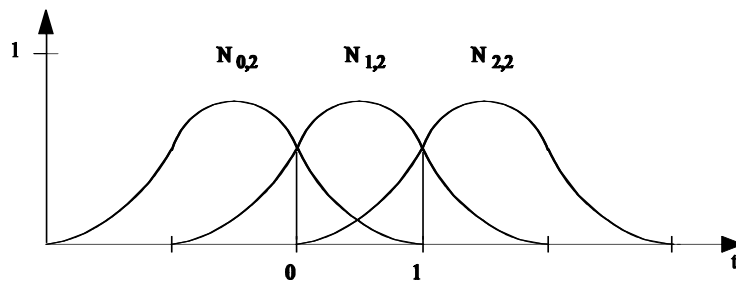$$N_{0,2}(t) = (1-t)^2, \quad N_{1,2}(t) = 2\,t(1-t), \quad N_{2,2}(t) = t^2 \qquad (7)$$

Fig. 3. Function $N_{i,2}$

For $n = 3$ and $T = \{0, 0, 0, 0, 1, 1, 1, 1\}$ the nonzero B-spline basis functions have the following form (Fig. 4).

$$N_{0,3}(t) = (1-t)^3 \ , \quad N_{1,3}(t) = 3\,t(1-t)^2$$
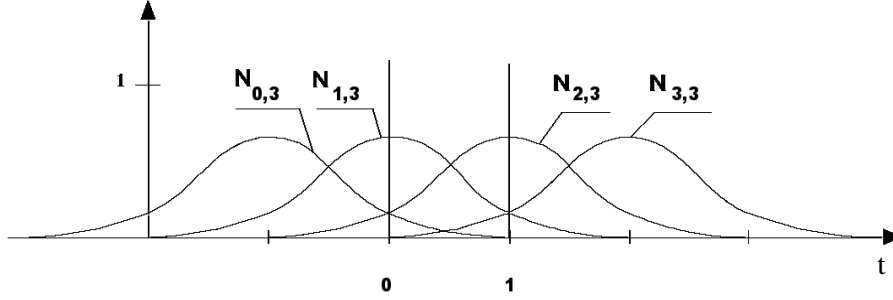$$N_{2,3}(t) = 3\,t^2(1-t), \quad N_{3,3}(t) = t^3 \tag{8}$$

Fig. 4. Function $N_{i,\,3}$

2. B-spline curves

A n-th degree B-spline curve is defined as follows

$$C(t) = \sum_{j=0}^{r} N_{j,n}(t)\,\mathbf{P}_j, \qquad a \le t \le b \tag{9}$$

where $\mathbf{P}_j$ are the control points, $N_{j,n}(t)$ are the B-spline basis functions (c.f. Equation (1)) defined for the following set of nodes

$$T = \left\{ a, ..., a, \ t_{n+1}, \ t_{n+2},...,\ t_{m-(n+1)}, b, ..., b \right\} \tag{10}$$

at the same time the values a and b appear $n+1$ times. The number of control points equals $r+1$ and corresponds to the number of nonzero basis functions.

It should be pointed out that the degree n of curve, number of control points $r+1$ and number of nodes $m+1$ are related by formula: $r = m - (n+1)$.

For example, if $n = 3$, $T = \{0, 0, 0, 0, 1, 1, 1, 1\}$ ($m = 7$) then the number of control points equals $r+1 = 4$ and the B-spline curve has the following form

$$C(t) = N_{0,3}\,P_0 + N_{1,3}\,P_1 + N_{2,3}\,P_2 + N_{3,3}\,P_3 \tag{11}$$

where the functions $N_{i,\,3}$, $i = 0, 1, 2, 3$ are described by formula (8).

In Figure 5 the B-spline curves for the control points a) $\mathbf{P}_0 = (0, 0)$, $\mathbf{P}_1 = (1, 0)$, $\mathbf{P}_2 = (0.75, 0.5)$, $\mathbf{P}_3 = (0, 1)$ and b) $\mathbf{P}_0 = (0, 0)$, $\mathbf{P}_1 = (0.75, 0.5)$, $\mathbf{P}_2 = (1, 0)$, $\mathbf{P}_3 = (0, 1)$ are shown.
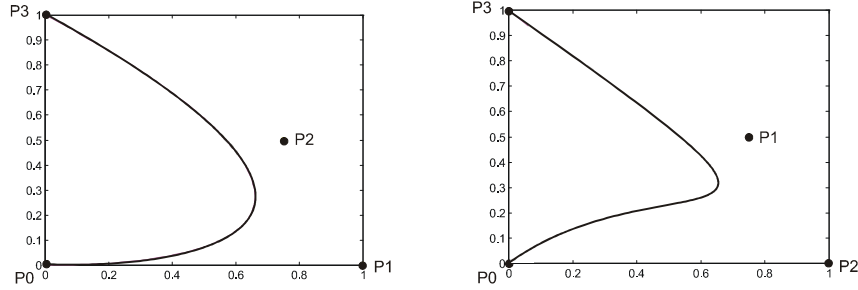

Fig. 5. B-spline curves for 4 control points and $T = \{0, 0, 0, 0, 1, 1, 1, 1\}$

3. NURBS curves

A n-th degree NURBS curve is defined as follows

$$C(t) = \frac{\displaystyle\sum_{j=0}^{r} N_{j,n}(t)\, w_j\, \mathbf{P}_j}{\displaystyle\sum_{k=0}^{r} N_{k,n}(t)\, w_k}, \qquad a \leq t \leq b \tag{12}$$

where $\mathbf{P}_j$ are the control points forming a control polygon, w_j are the weights and $N_{j,n}(t)$ are the B-spline basis functions defined for the set of nodes (10).

For example, for $n = 3$, $T = \{0, 0, 0, 0, 1, 1, 1, 1\}$ ($m = 7$) the number of control points equals $r + 1 = 4$ and the NURBS curve has the following form

$$C(t) = \frac{N_{0,3}\, w_0 \mathbf{P}_0 + N_{1,3}\, w_1 \mathbf{P}_1 + N_{2,3}\, w_2 \mathbf{P}_2 + N_{3,3}\, w_3 \mathbf{P}_3}{N_{0,3} w_0 + N_{1,3} w_1 + N_{2,3} w_2 + N_{3,3} w_3} \tag{13}$$
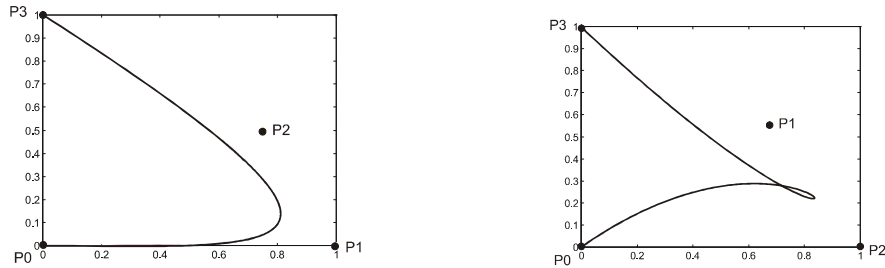

Fig. 6. NURBS curves for 4 control points and $T = \{0, 0, 0, 0, 1, 1, 1, 1\}$

In Figure 6 the NURBS curves for the control points marked in Figure 5 and the weights a) $w_0 = 1$, $w_1 = 3$, $w_2 = 1$, $w_3 = 1$ and b) $w_0 = 1$, $w_1 = 3$, $w_2 = 4$, $w_3 = 1$, are shown.

4. Application of NURBS curves for description of complex domain geometry

The following set of control points has been taken into account: $\mathbf{P}_0 = (0.005, 0.005)$, $\mathbf{P}_2 = (0.005, 0)$, $\mathbf{P}_3 = (0.01, 0)$, $\mathbf{P}_4 = (0.04, 0)$, $\mathbf{P}_5 = (0.04, 0)$, $\mathbf{P}_6 = (0.04, 0.025)$, $\mathbf{P}_7 = (0.015, 0.025)$, $\mathbf{P}_8 = (0.0025, 0.025)$, $\mathbf{P}_9 = (0, 0.02)$, $\mathbf{P}_{10} = (0.005, 0.015)$. The segments of boundary of the domain presented in Figure 7 are described by the following NURBS curves ($w_0 = w_1 = 1$, $w_2 = 2$)

$$C_1: \qquad C_1(t) = \frac{(1-t)^2 w_0 \mathrm{P}_0 + 2\,t(1-t)w_1 \mathrm{P}_1 + t^2 w_2 \mathrm{P}_2}{(1-t)^2 w_0 + 2\,t(1-t)w_1 + t^2 w_2}$$

$$C_2: \qquad C_2(t) = (1-t)\mathrm{P}_2 + t\mathrm{P}_3$$

$$C_3: \qquad C_3(t) = \frac{(1-t)^2 w_0 \mathrm{P}_3 + 2\,t(1-t)w_1 \mathrm{P}_4 + t^2 w_2 \mathrm{P}_5}{(1-t)^2 w_0 + 2\,t(1-t)w_1 + t^2 w_2}$$

$$C_4: \qquad C_4(t) = (1-t)\mathrm{P}_5 + t\mathrm{P}_6 \ , \qquad\qquad C_5: \qquad C_5(t) = (1-t)\mathrm{P}_6 + t\mathrm{P}_7$$

$$C_6: \qquad C_6(t) = \frac{(1-t)^2 w_0 \mathrm{P}_7 + 2t(1-t)w_1 \mathrm{P}_8 + t^2 w_2 \mathrm{P}_9}{(1-t)^2 w_0 + 2t(1-t)w_1 + t^2 w_2}$$

$$C_7: \qquad C_7(t) = (1-t)\mathrm{P}_9 + t\mathrm{P}_{10} \ , \qquad\qquad C_8: \qquad C_8(t) = (1-t)\mathrm{P}_{10} + t\mathrm{P}_0$$
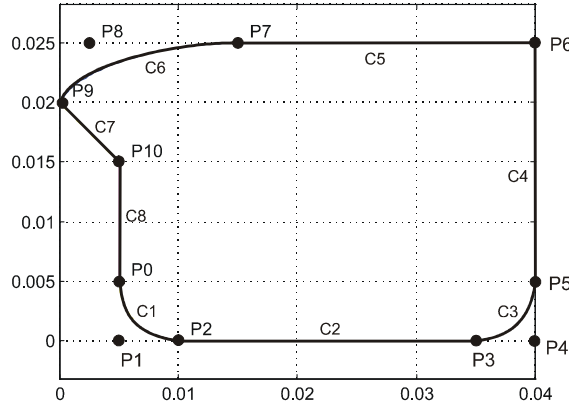

Fig. 7. Boundary of the domain described by NURBS curves

Summing up, proper choice of control points, their weights and degrees of NURBS curves allows to describe practically optional boundary of 2D domain.

References

[1] Piegl L., Tiller W., The NURBS Book, Springer, 1995.

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