APPLICATION OF THE DRBEM FOR NUMERICAL SOLUTION OF CATTANEO-VERNOTTE BIOHEAT TRANSFER EQUATION

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Abstract. The 1D problem of heat transfer in the tissue subjected to action of external heat source is considered. This phenomenon is described by Cattaneo-Vernotte equation supplemented by adequate boundary and initial conditions. To solve the problem formulated the dual reciprocity boundary element method (DRBEM) is applied. In the final part of the paper the examples of computations are shown.

1. Formulation of the problem

According to the newest opinions the heat conduction proceeding in the biological tissue domain should be described by the hyperbolic equation (Cattaneo-Vernotte equation [1, 2]) in order to take into account its nonhomogeneous inner structure. So the following bio-heat transfer equation is considered

$$c\left[\tau\frac{\partial^2 T(x,t)}{\partial t^2}+\frac{\partial T(x,t)}{\partial t}\right]=\lambda\frac{\partial^2 T(x,t)}{\partial x^2}+Q(x,t)+\tau\frac{\partial Q(x,t)}{\partial t} \tag{1}$$

where c, λ denote the volumetric specific heat and thermal conductivity of tissue, $Q(x, t)$ is the capacity of internal heat sources due to metabolism and blood perfusion, τ is the relaxation time (for biological tissue it is a value from the scope 20-35 s), T is the tissue temperature, x, t denote the spatial co-ordinates and time. The function $Q(x, t)$ is equal to

$$Q(x,t)=G_B\, c_B\left[T_B-T(x,t)\right]+Q_m \tag{2}$$

where G_B is the blood perfusion rate, c_B is the volumetric specific heat of blood, T_B is the artery temperature and Q_m is the metabolic heat source.

The equation (1) is supplemented by the boundary conditions

$$\begin{aligned} x=0:&\quad T(x,t)=T_{b1}(t)\\ x=L:&\quad T(x,t)=T_{b2}\end{aligned} \tag{3}$$

and initial ones

$$t=0:\ T(x,t)=T_0\ ,\quad \left.\frac{\partial T(x,t)}{\partial t}\right|_{t=0}=0 \tag{4}$$

where $T_{b1}(t)$, T_{b2} are known boundary temperatures and T_0 is known initial temperature of biological tissue.

It should be pointed out that for $\tau = 0$ the equation (1) reduces to the well-known Pennes bioheat equation.

2. Boundary element method

Taking into account formula (2) the equation (1) can be written in the form

$$\begin{gathered} c\left[\tau\frac{\partial^2 T(x,t)}{\partial t^2}+\frac{\partial T(x,t)}{\partial t}\right]= \\ \lambda\frac{\partial^2 T(x,t)}{\partial x^2}+Q_m+G_B\,c_B\left[T_B-T(x,t)\right]-\tau G_B\,c_B\frac{\partial T(x,t)}{\partial t} \end{gathered} \tag{5}$$

or

$$\begin{gathered} \lambda\frac{\partial^2 T(x,t)}{\partial x^2}-(c+\tau G_B\,c_B)\frac{\partial T(x,t)}{\partial t}-c\,\tau\frac{\partial^2 T(x,t)}{\partial t^2}+ \\ G_B\,c_B\left[T_B-T(x,t)\right]+Q_m=0 \end{gathered} \tag{6}$$

For $t = t^f$ one has

$$\lambda\frac{\partial^2 T(x,t^f)}{\partial x^2}-S(x,t^f)=0 \tag{7}$$

where

$$\begin{gathered} S(x,t^f)=(c+\tau G_B\,c_B)\left.\frac{\partial T(x,t)}{\partial t}\right|_{t=t^f}+c\,\tau\left.\frac{\partial^2 T(x,t)}{\partial t^2}\right|_{t=t^f}- \\ G_B\,c_B\left[T_B-T(x,t^f)\right]-Q_m \end{gathered} \tag{8}$$

Application of the standard boundary element method leads to the following equation [3, 4]

$$T(\xi, t^f) + T^*(\xi, L)\, q(L, t^f) - T^*(\xi, 0)\, q(0, t^f) = \\ q^*(\xi, L)\, T(L, t^f) - q^*(\xi, 0)\, T(0, t^f) - \int_0^L S(x, t^f)\, T^*(\xi, x)\, \mathrm{d}x \tag{9}$$

where ξ is the observation point, $T^*(\xi, x)$ is the fundamental solution, $q(x, t^f) = -\lambda\, \partial\, T(x, t^f) / \partial\, x$ is the heat flux, $q^*(\xi, x) = -\lambda\, \partial\, T^*(\xi, x) / \partial\, x$ is the heat flux resulting from the fundamental solution.

For the problem considered the fundamental solution has following form

$$T^*(\xi, x) = \frac{1}{2\lambda}(L - |x - \xi|) \tag{10}$$

Heat flux resulting from the fundamental solution can be calculated analytically

$$q^*(\xi, x) = -\lambda \frac{\partial T^*(\xi, x)}{\partial x} = \frac{\operatorname{sgn}(x - \xi)}{2} \tag{11}$$

It should be pointed out that the function $T^*(\xi, x)$ fulfills the equation

$$\lambda \frac{\partial^2 T^*(\xi, x)}{\partial x^2} = -\delta(\xi, x) \tag{12}$$

where $\delta(\xi, x)$ is the Dirac function.

3. Dual reciprocity boundary element method

The solution of Cattaneo-Vernotte equation (6) written for time t^f (Equation (7)) can be expressed as a sum [5, 6]

$$T(x, t^f) = \hat{T}(x, t^f) + U(x, t^f) \tag{13}$$

where the first function is the solution of Laplace equation

$$\lambda \frac{\partial^2 \hat{T}(x, t^f)}{\partial x^2} = 0 \tag{14}$$

and $U(x, t^f)$ is a particular solution of Equation (7), this means

$$\lambda \frac{\partial^2 U(x,t)}{\partial x^2} - (c + \tau G_B c_B) \frac{\partial U(x,t)}{\partial t} - c\tau \frac{\partial^2 U(x,t)}{\partial t^2} + \\ G_B c_B \left[T_B - U(x, t)\right] + Q_m = 0 \tag{15}$$

From Equations (6), (13), (14), (15) results that

$$\left[(c+\tau G_B c_B)\frac{\partial U(x,t)}{\partial t}+c\tau\frac{\partial^2 U(x,t)}{\partial t^2}+G_B c_B U(x,t)\right]_{t=t^f} = \left[(c+\tau G_B c_B)\frac{\partial T(x,t)}{\partial t}+c\tau\frac{\partial^2 T(x,t)}{\partial t^2}+G_B c_B T(x,t)\right]_{t=t^f} \tag{16}$$

So, the source function (8) can be expressed as follows

$$S(x,t^f)=(c+\tau G_B c_B)\left.\frac{\partial U(x,t)}{\partial t}\right|_{t=t^f}+c\tau\left.\frac{\partial^2 U(x,t)}{\partial t^2}\right|_{t=t^f}- G_B c_B\left[T_B-U(x,t^f)\right]-Q_m \tag{17}$$

It is generally difficult to find the solution $U(x, t^f)$. In the dual reciprocity method, at first, the following approximation of function $S(x, t^f)$ (c.f. Equation (8)) is proposed [5]

$$S(x,t^f)=\sum_{k=1}^{K} a_k(t^f)P_k(x) \tag{18}$$

where $a_k(t^f)$ are unknown coefficients and $P_k(x)$ are approximating functions fulfilling the equations

$$P_k(x)=\lambda\frac{\partial^2 U_k(x,t^f)}{\partial x^2} \tag{19}$$

In Equation (18), K corresponds to the total number of nodes, where 2 is the number of boundary nodes and $K-2$ is the number of internal nodes.

The last integral in equation (9) can be expressed as follows

$$D=-\int_0^L S(x,t^f)T^*(\xi,x)\,\mathrm{d}x=-\int_0^L\sum_{k=1}^{K}\lambda\, a_k(t^f)\frac{\partial^2 U_k(x)}{\partial x^2}T^*(\xi,x)\,\mathrm{d}x \tag{20}$$

Integrating twice by parts with respect to x one obtains

$$D=-\sum_{k=1}^{K}a_k(t^f)\int_0^L\left[\lambda\frac{\partial^2 T^*(\xi,x)}{\partial x^2}\right]U_k(x)\,dx- \sum_{k=1}^{K}a_k(t^f)\left[\lambda T^*(\xi,x)\frac{\partial U_k(x)}{\partial x}-\lambda U_k(x)\frac{\partial T^*(\xi,x)}{\partial x}\right]_{x=0}^{x=L} \tag{21}$$

Taking into account the property (12) one has

$$D = \sum_{k=1}^{K} a_k(t^f)\Big[U_k(\xi) + T^*(\xi,\, x)\, W_k(x) - U_k(x)\, q^*(\xi,\, x)\Big]_{x=0}^{x=L} \tag{22}$$

where

$$W_k(x) = -\lambda \frac{\partial U_k(x)}{\partial x} \tag{23}$$

Finally, the Equation (9) can be written in the form

$$\begin{gathered} T(\xi,\, t^f) + T^*(\xi,\, L)\, q(L,\, t^f) - T^*(\xi,\, 0)\, q(0,\, t^f) = q^*(\xi,\, L) T(L,\, t^f) \\ - q^*(\xi,\, 0) T(0,\, t^f) + \sum_{k=1}^{K} a_k(t^f)\Big[U_k(\xi) + T^*(\xi,\, L) W_k(L) - \\ T^*(\xi,\, 0) W_k(0) - q^*(\xi,\, L) U_k(L) + q^*(\xi,\, 0) U_k(0)\Big] \end{gathered} \tag{24}$$

We define [5]

$$U_k(x) = \frac{|x_k - x|^2}{4} + \frac{|x_k - x|^3}{9} \tag{25}$$

from which results that

$$W_k(x) = \lambda\,(x_k - x)\left(\frac{1}{2} + \frac{|x_k - x|}{3}\right) \tag{26}$$

On the basis of (25) the functions (19) are calculated

$$P_k(x) = \lambda\left(\frac{1}{2} + \frac{2|x_k - x|}{3}\right) \tag{27}$$

The following approximations of time derivatives appearing in formula (8) can be applied

$$\left.\frac{\partial T(x,\, t)}{\partial t}\right|_{t=t^f} = \frac{T(x,\, t^f) - T(x,\, t^{f-1})}{\Delta t} \tag{28}$$

$$\left.\frac{\partial^2 T(x,\, t)}{\partial t^2}\right|_{t=t^f} = \frac{T(x,\, t^f) - 2T(x,\, t^{f-1}) + T(x,\, t^{f-2})}{(\Delta t)^2} \tag{29}$$

So, we have (c.f. Equation (18))

$$
\begin{gathered}
(c+\tau G_B c_B)\frac{T(x,t^f)-T(x,t^{f-1})}{\Delta t}+\\
c\tau\frac{T(x,t^f)-2T(x,t^{f-1})+T(x,t^{f-2})}{(\Delta t)^2}+\\
G_B c_B T(x,t^f)-G_B c_B T_B-Q_m=\sum_{k=1}^{K}a_k(t^f)P_k(x)
\end{gathered}
\tag{30}
$$

or

$$
AT(x,t^f)+BT(x,t^{f-1})+CT(x,t^{f-2})+D=\sum_{k=1}^{K}a_k(t^f)P_k(x)
\tag{31}
$$

where

$$
\begin{gathered}
A=\frac{c+\tau G_B c_B}{\Delta t}+\frac{c\tau}{(\Delta t)^2}+G_B c_B\ ,\quad B=-\left(\frac{c+\tau G_B c_B}{\Delta t}+\frac{2c\tau}{(\Delta t)^2}\right)\\
C=\frac{c\tau}{(\Delta t)^2}\ ,\quad D=-G_B c_B T_B-Q_m
\end{gathered}
\tag{32}
$$

Let $x_1 = 0$, $x_2 = L$, and x_i are the internal nodes, i = 3, 4,..., K. For these nodes the following equations are obtained

$$
AT_i^f+BT_i^{f-1}+CT_i^{f-2}+D=\sum_{k=1}^{K}a_k^f P_k(x_i)\ ,\quad i=1,2,\ldots,K
\tag{33}
$$

This system of equations can be written in the matrix form

$$
\begin{bmatrix}
AT_1^f+BT_1^{f-1}+CT_1^{f-2}+D\\
AT_2^f+BT_2^{f-1}+CT_2^{f-2}+D\\
AT_3^f+BT_3^{f-1}+CT_3^{f-2}+D\\
\ldots\\
AT_K^f+BT_K^{f-1}+CT_K^{f-2}+D
\end{bmatrix}
=
\begin{bmatrix}
P_1(x_1) & P_2(x_1) & \ldots & P_K(x_1)\\
P_1(x_2) & P_2(x_2) & \ldots & P_K(x_2)\\
P_1(x_3) & P_2(x_3) & \ldots & P_K(x_3)\\
\ldots & \ldots & \ldots & \ldots\\
P_1(x_K) & P_2(x_K) & \ldots & P_K(x_K)
\end{bmatrix}
\begin{bmatrix}
a_1^f\\ a_2^f\\ a_3^f\\ \ldots\\ a_K^f
\end{bmatrix}
\tag{34}
$$

or

$$
A\mathbf{T}^f+B\mathbf{T}^{f-1}+C\mathbf{T}^{f-2}+D=\mathbf{P}\mathbf{a}^f
\tag{35}
$$

from which results that

$$
\mathbf{a}^f=\mathbf{P}^{-1}(A\mathbf{T}^f+B\mathbf{T}^{f-1}+C\mathbf{T}^{f-2}+D)
\tag{36}
$$

The Equation (24) for the nodes x_i, $i = 1, 2, ..., K$ can be expressed in the form

$$
\begin{aligned}
&T(x_i, t^f) + G_{i1}\, q(0, t^f) + G_{i2}\, q(L, t^f) = H_{i1}\, T(0, t^f) + H_{i2}\, T(L, t^f) + \\
&\sum_{k=1}^{K} a_k(t^f) \left[U_k(x_i) + G_{i1} W_k(0) + G_{i2} W_k(L) + H_{i1} U_k(0) + H_{i2} U_k(L) \right]
\end{aligned}
\tag{37}
$$

or

$$
\begin{aligned}
&T_i^f + G_{i1}\, q_1^f + G_{i2}\, q_2^f = H_{i1}\, T_1^f + H_{i2}\, T_2^f + \\
&\sum_{k=1}^{K} a_k \left[U_k(x_i) + G_{i1} W_k(x_1) + G_{i2} W_k(x_2) - H_{i1} U_k(x_1) - H_{i2} U_k(x_2) \right]
\end{aligned}
\tag{38}
$$

where

$$
G_{i1} = -\frac{L - x_i}{2\lambda}, \quad G_{i2} = \frac{x_i}{2\lambda}, \quad H_{i1} = H_{i2} = \frac{1}{2}
\tag{39}
$$

The system of Equations (38) can be written in the matrix form

$$
\mathbf{G}\,\mathbf{q}^f = \mathbf{H}\,\mathbf{T}^f + (\mathbf{G}\,\mathbf{W} - \mathbf{H}\,\mathbf{U})\,\mathbf{a}^f
\tag{40}
$$

where

$$
\mathbf{G} = \begin{bmatrix}
G_{11} & G_{12} & 0 & \dots & 0 \\
G_{21} & G_{22} & 0 & \dots & 0 \\
G_{31} & G_{32} & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots \\
G_{K1} & G_{K2} & 0 & \dots & 0
\end{bmatrix}
\tag{41}
$$

$$
\mathbf{H} = \begin{bmatrix}
H_{11} - 1 & H_{12} & 0 & \dots & 0 \\
H_{21} & H_{22} - 1 & 0 & \dots & 0 \\
H_{31} & H_{32} & -1 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots \\
H_{K1} & H_{K2} & 0 & \dots & -1
\end{bmatrix}
\tag{42}
$$

and

$$
\mathbf{U} = \begin{bmatrix}
U_1(x_1) & U_2(x_1) & U_3(x_1) & \dots & U_K(x_1) \\
U_1(x_2) & U_2(x_2) & U_3(x_2) & \dots & U_K(x_2) \\
U_1(x_3) & U_2(x_3) & U_3(x_3) & \dots & U_K(x_3) \\
\dots & \dots & \dots & \dots & \dots \\
U_1(x_K) & U_2(x_K) & U_3(x_K) & \dots & U_K(x_K)
\end{bmatrix}
\tag{43}
$$

$$\mathbf{W}=\begin{bmatrix} W_1(x_1) & W_2(x_1) & W_3(x_1) & \dots & W_K(x_1) \\ W_1(x_2) & W_2(x_2) & W_3(x_2) & \dots & W_K(x_2) \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{44}$$

while

$$\mathbf{T}^f=\begin{bmatrix} T_1^f \\ T_2^f \\ T_3^f \\ \dots \\ T_K^f \end{bmatrix}, \quad \mathbf{q}^f=\begin{bmatrix} q_1^f \\ q_2^f \\ 0 \\ \dots \\ 0 \end{bmatrix}, \quad \mathbf{a}^f=\begin{bmatrix} a_1^f \\ a_2^f \\ a_3^f \\ \dots \\ a_K^f \end{bmatrix} \tag{45}$$

Putting (36) into (40) one has

$$\mathbf{G}\,\mathbf{q}^f=\mathbf{H}\,\mathbf{T}^f+\mathbf{M}(A\mathbf{T}^f+B\mathbf{T}^{f-1}+C\,\mathbf{T}^{f-2}+D) \tag{46}$$

where

$$\mathbf{M}=(\mathbf{G}\,\mathbf{W}-\mathbf{H}\,\mathbf{U})\mathbf{P}^{-1} \tag{47}$$

The boundary conditions (3) should be introduced to the system of equations (46) and next this system can be solved. A start point of numerical simulation process results from the initial conditions (4), in particular $T_i^0 = T_i^1 = T_0$ for $i = 3, 4, \dots K$.

4. Results of computations

The layer of tissue (L = 1 cm) limited by the skin surface and conventionally assumed internal one is considered. The following input data have been introduced λ = 0.5 W/(mK), c = 4.2 MJ/(m^3 K), c_B = 3.9962 MJ/(m^3 K), G_B = 0.002 1/s, T_B = 37°C, Q_m = 420 W/m^3, τ = 35 s, T_0 = 37°C. The boundary condition on the skin surface has been assumed in the form $T_{b1}(t) = 37 + 0.25t$, while for internal surface $x = L$: T_{b2} = 37°C.

The problem has been solved by means of the DRBEM under the assumption that in the interior of domain 99 internal points have been distinguished and Δt = 0.5. In Figure 1 the temperature distribution in the tissue for times 10 and 20 s for τ = 35 s (Cattaneo-Vernotte model) and τ = 0 (Pennes model) is shown. Figure 2 illustrates the heating curves at two points selected from the domain considered.

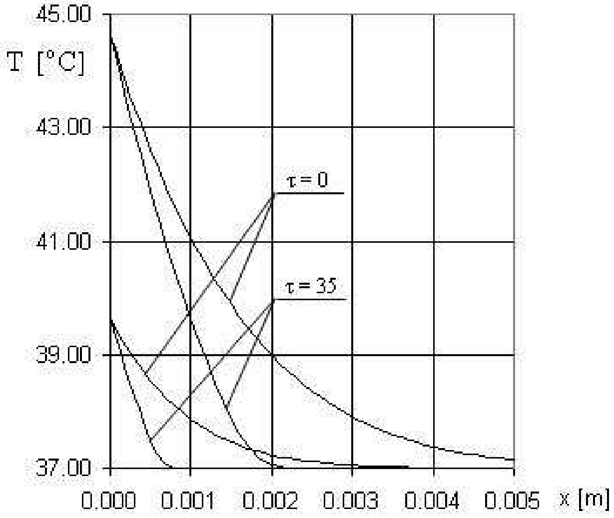

Fig. 1. Comparison of Cattaneo-Vernotte and Pennes models - temperature distribution

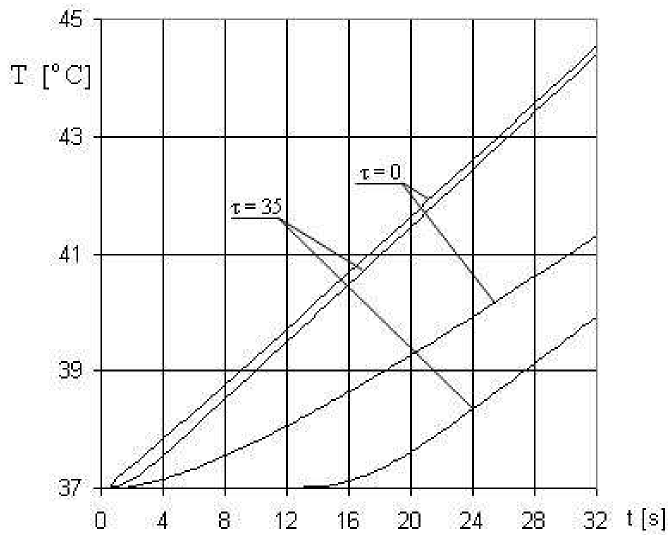

Fig. 2. Comparison of Cattaneo-Vernotte and Pennes models - heating curves for x = 0.0001 m and x = 0.001 m

The numerical solution of Cattaneo-Vernotte equation in comparison with the Pennes one leads to the visible different results. Introduction of relaxation time causes that the process proceeds slower.

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