ON THE MODELLING OF A CERTAIN QUASI-LINEAR HEAT CONDUCTION PROBLEM IN A TWO-PHASED LAMINATED MEDIUM

Urszula Siedlecka[1], Czesław Woźniak[2]

[1] *Institute of Mathematics, Czestochowa University of Technology, Poland*
urszulas@imi.pcz.pl
[2] *Department of Structural Mechanics, Technical University of Lodz, Poland*

Abstract. The aim of the contribution is to obtain a macroscopic model equation for the quasi-linear heat conduction in a two phased-laminated medium. The analysis will be based on the tolerance averaging technique.

1. The quasi-linear heat conduction

Problem of quasi-linear heat conduction is not new. Some informations on this subject one can find in [1]. The well-known quasi-linear equation of the heat conduction

$$\frac{\partial}{\partial x}\left(k\frac{\partial \vartheta}{\partial x}\right) - \rho c\frac{\partial \vartheta}{\partial t} = 0 \tag{1}$$

where heat conductivity k and specific heat c depend on temperature ϑ and ρ is a mass density, can be reduced to the form

$$\frac{\partial^2 \Theta}{\partial x^2} - \frac{1}{\kappa}\frac{\partial \Theta}{\partial t} = 0 \tag{2}$$

by means of

$$\Theta = \int_0^{\vartheta} \frac{k}{k_0} d\vartheta \tag{3}$$

where $\kappa = k/\rho c$ is a function of argument Θ.

This situation takes place only if the heat conductor is homogeneous. For a periodically heterogeneous conductor the heat conduction problems can be investigated by using the asymptotic homogenization procedure, cf. Artole and Duvaut [2]. However, the homogenized equations are independent of the microstructure size. To remove this drawback an alternative approach to the macro--modelling of heat conduction was proposed, known as the tolerance averaging technique [3]. This technique was applied to the heat conduction analysis in [4-7].

The aforementioned contributions were restricted to the analysis of linear problems. In contrast to the above papers, in this contribution we are to apply the tolerance averaging technique to the investigation of a certain special quasi-linear heat conduction problem.

2. Formulation of the problem

The object of considerations is a two-phased micro-periodic laminate a fragment of which is shown in Figure 1. Let us assume that the laminae are homogeneous and isotropic. Moreover, the heat conduction coefficients k', k'' in pertinent laminae depend on the increment of temperature θ by means of

$$\begin{aligned} k' &= k'_0(1+\delta'\theta) \\ k'' &= k''_0(1+\delta''\theta) \end{aligned} \tag{4}$$

Here, k'_0, k''_0 are positive and δ, δ' are non-negative material constants. Hence, material properties of the conductor are uniquely characterized by functions $k_0(x_1)$, $\delta(x_1)$, $\rho(x_1)$, $c(x_1)$ which are piecewise constant.

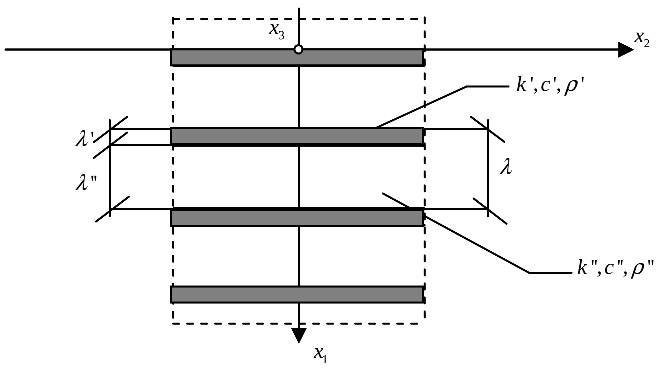

Fig. 1. A two-phased laminated medium

It is assumed that period λ (c.f. Fig. 1) is sufficiently small when compared to the smallest characteristic length dimension of the conductor. By c', c'' and ρ', ρ'' we denote a specific heat and a mass density in pertinent laminaes, respectively. It has to be emphasized that the temperature θ in formula (4) is restricted by conditions $k' > 0$ and $k'' > 0$.

The aim of this contribution is to derive the macroscopic mathematical model related to the problem described by equation

$$\partial_i \left(k \partial_i \theta \right) - \rho c \dot{\theta} = 0 \tag{5}$$

under assumption (4). To this end the tolerance averaging method will be used [3].

3. Analysis

As a tool of macro-modelling of equation (5) the tolerance averaging technique will be taken into account. We denote $\mathbf{x} = \left(x_1, x_2, x_3 \right)$ as cartesian orthogonal coordinates in space and t as a time coordinate. Using this technique we introduce the micro-macro-decomposition

$$\theta\left(\mathbf{x},t\right) = \vartheta\left(\mathbf{x},t\right) + h\left(x_1\right)\psi\left(\mathbf{x},t\right) \tag{6}$$

where $h(x_1)$ is the saw-like shape function [3]. The basic unknowns are: averaged temperature $\vartheta = \vartheta\left(\mathbf{x},t\right)$ and temperature fluctuation amplitude $\psi = \psi\left(\mathbf{x},t\right)$.

After tolerance averaging of equation (5) we obtain the system of equations for ϑ and ψ in the form

$$\begin{aligned} &\partial_i \left\langle k \right\rangle \partial_i \vartheta + \left\langle k \partial_1 h \right\rangle \partial_1 \psi - \left\langle \rho c \right\rangle \dot{\vartheta} = 0 \\ &\lambda^2 \left\langle k \right\rangle \partial_\alpha \partial_\alpha \psi - \left\langle k \left(\partial_1 h \right)^2 \right\rangle \psi - \left\langle k \partial_1 h \right\rangle \partial_1 \vartheta - \lambda^2 \left\langle \rho c \right\rangle \dot{\psi} = 0 \end{aligned} \tag{7}$$

where $i = 1,2,3$ and $\alpha = 2,3$. At the same time we have

$$k = k_0\left(x_1\right)\Big(1 + \delta\left(x_1\right)\big(\vartheta + h\left(x_1\right)\psi\big)\Big) \tag{8}$$

It can be proved that after substituting (8) to (7), coefficients in model equations are independent of temperature fluctuation amplitude ψ.

4. Final result

The main result of the above analysis is that the quasi-linearity of equations (7) is imposed only on averaged temperature ϑ but the problem is linear with respect to the temperature fluctuation amplitude ψ. The obtained model equations will be applied to the investigation of special heat conduction problems in a separate contribution.

References

[1] Carslaw H.S., Jaeger J.C., Conduction of heat in solids, Oxford at the Clarendon Press, 1959.

[2] Artola M., Duvaut G., Annales de la Faculte des Sciences de Toulouse, 1982.

[3] Woźniak C., Wierzbicki E., Averaging technique in thermomechanics of composite solids, Wydawnictwo Politechniki Częstochowskiej, Częstochowa 2000.

[4] Łaciński Ł., Numerical verification of two mathematical models for the heat transfer in a laminated rigid conductor, Journal of Theoretical and Applied Mechanics 2005, 43, 2, 367-384.

[5] Łaciński Ł., Rychlewska J., Szymczyk J., Woźniak C., A contribution to the modeling of nonstationary processes in functionally graded laminates, Prace Naukowe Instytutu Matematyki i Informatyki Politechniki Częstochowskiej 2006, 83-95.

[6] Łaciński Ł., Woźniak C., Asymptotic models of the heat transfer in laminated conductors, EJPAU 2006, 9(2), #25.

[7] Szymczyk J., Woźniak C., On the certain approach to the hyperbolic heat propagation in a periodically laminated medium, Prace Naukowe Instytutu Matematyki i Informatyki Politechniki Częstochowskiej 2004, 209-215.