FREE VIBRATION OF STEPPED TIMOSHENKO BEAMS WITH ATTACHEMENTS

Izabela Zamorska

Institute of Mathematics, Czestochowa University of Technology, Poland
iz@imi.pcz.pl

Abstract. In this paper is presented a solution of the free vibration problem of multi-stepped Timoshenko beam with attachments. The Green's functions method is used to obtain the solution in an analytical form. The presented approach can be used for the vibration analysis of beams consisting of an arbitrary number of segments and an arbitrary number of attached discrete elements.

Introduction

Free vibrations of a uniform and non-uniform beam according to the Timoshenko theory are the subject of research of many authors, for example the papers [1-6] are devoted to these vibration problems.

The equations of motion for a uniform Timoshenko beam is derived in [1].

To find a solution of vibration problem of systems whose main elements are beams a Green's function method was used in [2-4]. In a paper [2] the exact solution of uniform beam was obtained. The author of [3] and [4] presented an application of Green's function in vibration problem of system consisting of a uniform Timoshenko beam and multi-mass oscillators or other attached discrete elements [3, 4] and in vibration problem of a multi-stepped beam without any attachments [4].

The solution of the free vibration problem of the uniform Timoshenko beam with added discrete elements obtained by using the Lagrange multiplier is presented in [5]. In paper [6] author presents an approximation of the non-homogeneous beam with wariable cross-section by a number of homogeneous stepped beams with a constant cross-section.

The presented paper deals with the free vibration problem of a system consisting of a stepped Timoshenko beam and attached discrete elements. Formulation and solution to the problem take into account arbitrary finite number of uniform beams composing the stepped beam and an arbitrary number of additional elements. The analytical solution of considered problem is found with the use of the Green's functions method.

1. Formulation of the problem

The subject of this consideration is a free vibration problem of a stepped beam length $L=\sum_{i=1}^{n} L_i$ with n-segments. At points $x_{ij}\in\langle 0,x_i\rangle$ i = 1,2,…,n, $j=1,...,n_j$, of the beam discrete elements are attached (Fig. 1).

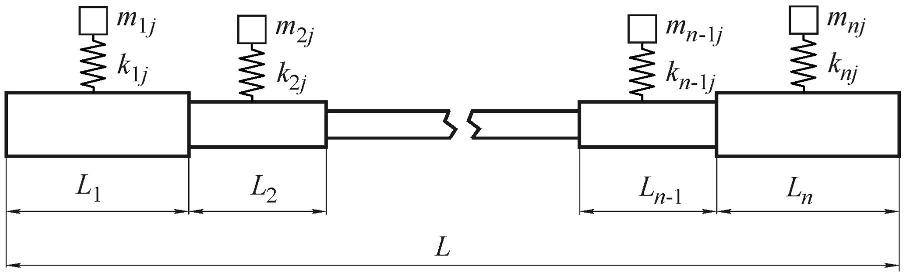

Fig. 1. A sketch of a non-uniform beam with attached discrete (spring-mass) elements

Free vibration of the beam is governed by the following Timoshenko equations [3]:

$$k_i A_i G_i\left(\frac{\partial^2 w_i}{\partial x_i^2}-\frac{\partial \varphi_i}{\partial x_i}\right)-\rho_i A_i\frac{\partial^2 w_i}{\partial t^2}=$$

$$=\tilde{\mathbf{M}}_{i1}\left[w_i,\varphi_i\right]+S_{i-1}(t)\,\delta(x_i)+S_i(t)\,\delta(x_i-L_i)$$

$$E_i I_i\frac{\partial^2 \varphi_i}{\partial x_i^2}+k_i A_i G_i\left(\frac{\partial w_i}{\partial x_i}-\varphi_i\right)-\rho_i I_i\frac{\partial^2 \varphi_i}{\partial t^2}=$$

$$=\tilde{\mathbf{M}}_{i2}\left[w_i,\varphi_i\right]+M_{i-1}(t)\,\delta(x_i)+M_i(t)\,\delta(x_i-L_i)$$

$$i = 1, 2,..., n\text{-}1;\; S_0 = M_0 = 0,\; x_i\in\left[0,L_i\right] \tag{1a}$$

$$k_n A_n G_n\left(\frac{\partial^2 w_n}{\partial x_n^2}-\frac{\partial \varphi_n}{\partial x_n}\right)-\rho_n A_n\frac{\partial^2 w_n}{\partial t_n^2}=\tilde{\mathbf{M}}_{1n}\left[w_n,\varphi_n\right]+S_{n-1}\delta(x_n)$$

$$E_n I_n\frac{\partial^2 \varphi_n}{\partial x_n^2}+k_n A_n G_n\left(\frac{\partial w_n}{\partial x_n}-\varphi_n\right)-\rho_n I_n\frac{\partial^2 \varphi_n}{\partial t_n^2}=\tilde{\mathbf{M}}_{2n}\left[w_n,\varphi_n\right]+M_{n-1}\delta(x_n)$$

$$x_n\in\left[0,L_n\right] \tag{1b}$$

where w_i is a deflection, φ_i is a bending slope, k_i is a factor depending on the shape of the cross-section, A_i is the cross-section area, G_i is a modulus of rigidity, E_i is a modulus of elasticity, I_i is a moment of inertia, ρ_i is the mass density of the beam material for every i = 1,2,...,n and $\delta(\,)$ denotes a Dirac delta function.

Linear differential operators: $\tilde{\mathbf{M}}_{1i}\left[w_i,\varphi_i\right]$, $\tilde{\mathbf{M}}_{2i}\left[w_i,\varphi_i\right]$, for i =1, 2, ..., n, depend on the attached discrete elements.

The functions w_i, φ_i (i = 1,n) satisfy homogeneous boundary conditions symbolically written in the form:

$$\overline{\mathbf{B}}_0\left[w_1,\varphi_1\right]\Big|_{x_1=0}=0\,,\qquad \overline{\mathbf{B}}_1\left[w_n,\varphi_n\right]\Big|_{x_n=L_n}=0 \tag{2}$$

Moreover, the continuity conditions at beam points $x_i = L_i$

$$w_i\Big|_{x_i=L_i}=w_{i+1}\Big|_{x_i=0}\,,\qquad \varphi_i\Big|_{x_i=L_i}=\varphi_{i+1}\Big|_{x_i=0}\quad i=1,2,\ldots,n\text{-}1 \tag{3}$$

are satisfied.

Assuming in the equations (1)-(3) that: $w(x_i,t)=\overline{W}(x_i)e^{i\omega t}$, $\varphi(x_i,t)=\overline{\psi}(x_i)e^{i\omega t}$, $S_i(t)=\overline{S}_i e^{i\omega t}$, $M_i(t)=\overline{M}_i e^{i\omega t}$, $\overline{S}_0=\overline{M}_0=0$, the following equations are obtained:

$$k_iA_iG_i\left(\overline{W}_i''-\overline{\psi}_i'\right)-\rho_i\omega^2A_i\overline{W}_i=\tilde{\mathbf{M}}_{1i}\left[\overline{W}_i,\overline{\psi}_i\right]+\overline{S}_{i-1}\delta\left(x_i\right)+\overline{S}_i\delta\left(x_i-L_i\right)$$

$$E_iI_i\overline{\psi}_i''+k_iA_iG_i\left(\overline{W}_i'-\overline{\psi}_i\right)-\rho_i\omega I_i\overline{\psi}_i=\tilde{\mathbf{M}}_{2i}\left[\overline{W}_i,\overline{\psi}_i\right]+\overline{M}_{i-1}\delta\left(x_i\right)+\overline{M}_i\delta\left(x_i-L_i\right)$$

$$i=1,2,\ldots,n\text{-}1 \tag{4a}$$

$$k_nA_nG_n\left(\overline{W}_n''-\overline{\psi}_n'\right)-\rho_n\omega^2A_n\overline{W}_n=\tilde{\mathbf{M}}_{1n}\left[\overline{W}_n,\overline{\psi}_n\right]+\overline{S}_{n-1}\delta\left(x_n\right)$$

$$E_nI_n\overline{\psi}_n''+k_nA_nG_n\left(\overline{W}_n'-\overline{\psi}_n\right)-\rho_n\omega^2I_n\overline{\psi}_n=\tilde{\mathbf{M}}_{2n}\left[\overline{W}_n,\overline{\psi}_n\right]+\overline{M}_{n-1}\delta\left(x_n\right) \tag{4b}$$

$$\overline{\mathbf{B}}_0\left[\overline{W}_1,\overline{\psi}_1\right]\Big|_{x_i=0}=0\,,\qquad \overline{\mathbf{B}}_1\left[\overline{W}_n,\overline{\psi}_n\right]\Big|_{x_i=L_n}=0 \tag{5}$$

$$\overline{W}_i\left(L_i\right)=\overline{W}_{i+1}(0)\,,\qquad \overline{\psi}_i\left(L_i\right)=\overline{\psi}_{i+1}(0)\qquad i=1,2,\ldots,n\text{-}1 \tag{6}$$

Introducing non-dimensional co-ordinates $\xi_i=\dfrac{x_i}{L_i}$, $W_i(\xi_i)=\dfrac{\overline{W}_i(x_i)}{L_i}$ and dimensionless values: $\alpha_i=\dfrac{L_{i+1}}{L_i}$, $\beta_i=\dfrac{k_{i-1}G_{i-1}A_{i-1}}{k_iG_iA_i}$, $\Omega_i^4=\dfrac{\rho_iA_iL_i\omega^2}{E_iI_i}$, $s_i{}^2=\dfrac{E_iI_i}{k_iA_iG_iL_i^2}$, $r_i^2=\dfrac{I_i}{A_iL_i^2}$, $S_i=\dfrac{\overline{S}_i}{k_iG_iA_i}$, $M_i=\dfrac{\overline{M}_i}{k_iG_iA_iL_i}$, i = 1,2,…,n, we obtain:

$$W_i''(\xi_i)-\psi_i'(\xi_i)+\Omega_i^4 s_i^2 W_i(\xi_i)=\tilde{\mathbf{M}}_{1i}\left[W_i(\xi_i),\psi_i(\xi_i)\right]+\beta_i S_{i-1}\delta(\xi_i)+S_i\delta(\xi_i-1)$$

$$\begin{aligned} &s_i^2\psi_i''(\xi_i)+W_i'(\xi_i)+\left(\Omega_i^4 s_i^2 r_i^2-1\right)\psi_i(\xi_i)= \\ &=\tilde{\mathbf{M}}_{2i}\left[W_i(\xi_i),\psi_i(\xi_i)\right]+\frac{\beta_i}{\alpha_{i-1}}M_{i-1}\delta(\xi_i)+M_i\delta(\xi_i-1)\end{aligned} \tag{7a}$$

$i=1, 2,...,n$-1

$$W_n''(\xi_n)-\psi_n'(\xi_n)+\Omega_n^4 s_n^2 W_n(\xi_n)=\tilde{\mathbf{M}}_{1n}\left[W_n(\xi_n),\psi_n(\xi_n)\right]+\beta_n S_{n-1}\delta(\xi_n)$$

$$s_n^2\psi_n''(\xi_n)+W_n'(\xi_n)+\left(\Omega_n^4 s_n^2 r_n^2-1\right)\psi_n(\xi_n)=\tilde{\mathbf{M}}_{2n}\left[W_n(\xi_n),\psi_n(\xi_n)\right]+\frac{\beta_n}{\alpha_{n-1}}M_{n-1}\delta(\xi_n) \tag{7b}$$

The boundary and continuity conditions (5), (6) may be written in the form:

$$\mathbf{B}_0\left[W_1,\psi_1\right]_{x_i=0}=0\,, \qquad \mathbf{B}_1\left[W_n,\psi_n\right]_{x_i=L_n}=0 \tag{8}$$

$$W_i(1)=\alpha_i W_{i+1}(0)\,, \qquad \psi_i(1)=\psi_{i+1}(0)\,, \qquad i=1,2,\ldots,n\text{-}1 \tag{9}$$

2. Solution of the problem

The solution of the considered free vibration problem will be obtained by using the Green’s function method [3, 4].

The Green’s matrix corresponding to the system of equations (7a,b) has the form:

$$\mathbf{G}_i(\xi_i,\eta_i)=\mathbf{G}_{0i}(\xi_i,\eta_i)+\mathbf{G}_{1i}(\xi_i,\eta_i)H(\xi_i-\eta_i)=\begin{bmatrix} g_1^i(\xi_i,\eta_i) & g_2^i(\xi_i,\eta_i) \\ g_3^i(\xi_i,\eta_i) & g_4^i(\xi_i,\eta_i)\end{bmatrix} \tag{10}$$

The matrixes $\boldsymbol{G}_{0i}$, $\boldsymbol{G}_{1i}$ are:

$$\mathbf{G}_{0i}(\xi_i,\eta_i)=\begin{bmatrix} g_{01}^i(\xi_i) & g_{02}^i(\xi_i) \\ g_{03}^i(\xi_i) & g_{04}^i(\xi_i)\end{bmatrix},\ \mathbf{G}_{1i}(\xi_i,\eta_i)=\begin{bmatrix} g_{11}^i(\xi_i-\eta_i) & g_{12}^i(\xi_i-\eta_i) \\ g_{13}^i(\xi_i-\eta_i) & g_{14}^i(\xi_i-\eta_i)\end{bmatrix}$$

where

$$g_s^i(\xi_i,\eta_i)=g_{0s}^i(\xi_i,)+g_{1s}^i(\xi_i-\eta_i)H(\xi_i-\eta_i) \qquad s=1,2,3,4 \tag{11}$$

and

$$g_{01}^{i}(\xi_i)=C_1\cosh\lambda_{1i}\xi_i+C_2\sinh\lambda_{1i}\xi_i+C_3\cos\lambda_{2i}\xi_i+C_4\sin\lambda_{2i}\xi_i$$

$$g_{02}^{i}(\xi_i)=C_1a_{1i}\sinh\lambda_{1i}\xi_i+C_2a_{1i}\cosh\lambda_{1i}\xi_i-C_3a_{2i}\sin\lambda_{2i}\xi_i+C_4a_{2i}\cos\lambda_{2i}\xi_i$$

$$g_{03}^{i}(\xi_i)=\overline{C}_1\cosh\lambda_{1i}\xi_i+\overline{C}_2\sinh\lambda_{1i}\xi_i+\overline{C}_3\cos\lambda_{2i}\xi_i+\overline{C}_4\sin\lambda_{2i}\xi_i$$

$$g_{04}^{i}(\xi_i)=\overline{C}_1a_{1i}\sinh\lambda_{1i}\xi_i+\overline{C}_2a_{2i}\cosh\lambda_{1i}\xi_i-\overline{C}_3a_{3i}\sin\lambda_{2i}\xi_i+\overline{C}_4a_{4i}\cos\lambda_{2i}\xi_i$$

$$g_{11}^{i}(\xi_i-\eta_i)=A_i\left[a_{1i}\sin\lambda_{2i}(\xi_i-\eta_i)-a_{2i}\sinh\lambda_{1i}(\xi_i-\eta_i)\right]$$

$$g_{13}^{i}(\xi_i-\eta_i)=a_{1i}a_{2i}A_i\left[\cos\lambda_{2i}(\xi_i-\eta_i)-\cosh\lambda_{1i}(\xi_i-\eta_i)\right]$$

$$g_{12}^{i}(\xi_i-\eta_i)=B_i\left[\cosh\lambda_{1i}(\xi_i-\eta_i)-\cos\lambda_{2i}(\xi_i-\eta_i)\right]$$

$$g_{14}^{i}(\xi_i-\eta_i)=B_i\left[a_{1i}\sinh\lambda_{1i}(\xi_i-\eta_i)+a_{2i}\sin\lambda_{2i}(\xi_i-\eta_i)\right] \qquad (12)$$

where $A_i=\dfrac{1}{a_{1i}\lambda_{2i}-a_{2i}\lambda_{1i}}$, $B_i=\dfrac{1}{s_i^2\left(a_{1i}\lambda_{1i}+a_{2i}\lambda_{2i}\right)}$, $a_{1i}=\dfrac{\sqrt{\Delta_i}-\kappa_{1i}}{2\lambda_{1i}}$, $a_{2i}=\dfrac{\sqrt{\Delta_i}+\kappa_{1i}}{2\lambda_{2i}}$

$$\lambda_{1i}=\sqrt{\frac{\sqrt{\Delta_i}-\kappa_{2i}}{2}},\quad \lambda_{2i}=\sqrt{\frac{\sqrt{\Delta_i}+\kappa_{2i}}{2}},\quad \sqrt{\Delta_i}=\Omega_i^2\sqrt{\kappa_{1i}+4}$$

$$\kappa_{1i}=\Omega_i^4\left(r_i^2-s_i^2\right),\quad \kappa_{2i}=\Omega_i^4\left(r_i^2+s_i^2\right)$$

By using the relationships (3.2.66)-(3.2.67) given in [3] and (7a,b) presented here, we obtain:

$$W_i(\xi_i)=\beta_i g_1^i(\xi_i,0)S_{i-1}+\frac{\beta_i}{\alpha_{i-1}}g_3^i(\xi_i,0)M_{i-1}+g_1^i(\xi_i,1)S_i+g_3^i(\xi_i,1)M_i+$$
$$+g_1^i(\xi_i,\zeta_{ij})\mathbf{M}_{1i}\left[W(\zeta_{ij})\right]+g_3^i(\xi_i,\zeta_{ij})\mathbf{M}_{2i}\left[\psi(\zeta_{ij})\right]$$

$$\psi_i(\xi_i)=\beta_i g_2^i(\xi_i,0)S_{i-1}+\frac{\beta_i}{\alpha_{i-1}}g_4^i(\xi_i,0)M_{i-1}+g_2^i(\xi_i,1)S_i+g_4^i(\xi_i,1)M_i+$$
$$+g_2^i(\xi_i,\zeta_{ij})\mathbf{M}_{1i}\left[W(\zeta_{ij})\right]+g_4^i(\xi_i,\zeta_{ij})\mathbf{M}_{2i}\left[\psi(\zeta_{ij})\right]$$

$$i=1,\ldots,n\text{-}1 \qquad (13a)$$

$$W_n(\xi_n) = \beta_n g_1^n(\xi_n,0) S_{n-1} + \frac{\beta_n}{\alpha_{n-1}} g_3^n(\xi_n,0) M_{n-1} + \\ + g_1^n(\xi_n,\zeta_{nj}) \mathbf{M}_{1n}\left[W(\zeta_{nj})\right] + g_3^n(\xi_n,\zeta_{nj}) \mathbf{M}_{2n}\left[\psi(\zeta_{nj})\right]$$

$$\psi_n(\xi_n) = \beta_n g_2^n(\xi_n,0) S_{n-1} + \frac{\beta_n}{\alpha_{n-1}} g_4^n(\xi_n,0) M_{n-1} + \\ + g_2^n(\xi_n,\zeta_{nj}) \mathbf{M}_{1n}\left[W(\zeta_{nj})\right] + g_4^n(\xi_n,\zeta_{nj}) \mathbf{M}_{2n}\left[\psi(\zeta_{nj})\right] \tag{13b}$$

Using equations (13a,b) in continuity conditions (9), the homogeneous set of equations with 2(n-1) unknown S_1, M_1, S_2, M_2,..., S_{n-1}, M_{n-1} are received. This set of equations supplement equations with unknown: W_1, Ψ_1, W_2, Ψ_2,..., W_N, Ψ_N, which are obtained from equations (13a,b) by substituting $\xi_i = \zeta_{ij}$ (i = 1,2,...,n, j = 1,2,...,n_j, $N = \sum_{j=1}^{n} n_j$). The non-trivial solution of a system of equations exist, if and only if the following condition is fulfilled:

$$\det\boldsymbol{A} = 0 \tag{14}$$

where $\boldsymbol{A}$ is a main matrix of system of equations.
Assuming that one spring-mass element is attached at i-th beam segment at $\xi_i = \zeta_{i1}$, the matrix $\boldsymbol{A}$ for n = 2, j = 1 has the following form:

$$\begin{bmatrix} g_1^1(1,1)-\alpha_1\beta_2 g_1^2(0,0) & g_3^1(1,1)-\beta_2 g_3^2(0,0) & g_1^1(1,\zeta_{11})\gamma_1 & 0 & -\alpha_1 g_1^2(0,\zeta_{21})\gamma_2 & 0 \\ g_2^1(1,1)-\beta_2 g_2^2(0,0) & g_4^1(1,1)-\frac{\beta_2}{\alpha_1} g_4^2(0,0) & g_2^1(1,\zeta_{11})\gamma_1 & 0 & -g_2^2(0,\zeta_{21})\gamma_2 & 0 \\ g_1^1(\zeta_{11},1) & g_3^1(\zeta_{11},1) & g_1^1(\zeta_{11},\zeta_{11})\gamma_1 - 1 & 0 & 0 & 0 \\ g_2^1(\zeta_{11},1) & g_4^1(\zeta_{11},1) & g_2^1(\zeta_{11},\zeta_{11})\gamma_1 & -1 & 0 & 0 \\ \beta_2 g_1^2(\zeta_{21},0) & \frac{\beta_2}{\alpha_1} g_3^2(\zeta_{21},0) & 0 & 0 & g_1^2(\zeta_{21},\zeta_{21})\gamma_2 - 1 & 0 \\ \beta_2 g_2^2(\zeta_{21},0) & \frac{\beta_2}{\alpha_1} g_4^2(\zeta_{21},0) & 0 & 0 & g_2^2(\zeta_{21},\zeta_{21})\gamma_2 & -1 \end{bmatrix} \tag{15}$$

where $\gamma_i = s_i^2\Omega_i^4 \dfrac{K_i M_i}{M_i\Omega_i^4 - K_i}$.

Equation (14) is the characteristic equation of the considered vibration problem which may be numerically solved with respect to non-dimensional frequency parameters Ω_i^4.

Conclusions

The solution for free vibrations of the stepped beam was obtained by using of the Green's function properties. The presented solution may be used to the numerical vibration analysis of a beam with an arbitrary number of segments and discrete elements. It's possible to approximate the non-homogeneous beam with variable cross-section by a number of homogeneous stepped beams with a constant cross-section.

References

[1] Bystrzycki J., Równania ruchu belek sprężystych, belka Timoshenki, Warszawska Drukarnia Naukowa, Warszawa 1977.

[2] Lueschen G.G.G., Bergman L.A., McFarland D.M., Green's Functions for uniform Timoshenko beams, Journal of Sound and Vibration 1996, 194(1), 93-102.

[3] Kukla S., Dynamiczne funkcje Greena w analizie drgań własnych ciągłych i dyskretno-ciągłych układów mechanicznych, Monografie nr 64, Częstochowa 1999.

[4] Kukla S., Application of Green functions in frequency analysis of Timoshenko beams with oscillators, Journal of Sound and Vibration 1997, 205(3), 355-363.

[5] Posiadała B., Free vibrations of uniform Timoshenko beams with attachments, Journal of Sound and Vibration 1997, 359-369.

[6] Tong X., Tabarrok B., Yeh K.Y., Vibration analysis of Timoshenko beams with non-homogeneity and varying cross-section, Journal of Sound and Vibration 1995, 186(5), 821-835.