ALGEBRAIC CURVES IN THE COMPLEX PLANE

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Introduction and previous results

An algebraic curve C in $\mathbb{C}^2$ is given by an algebraic equation

$$f(x,y)=0$$

From the geometrical point of view it is a Riemann surface. This surface has following invariants: geometrical genus, singular points and punctures (which correspond to points at infinity). Here under the *geometrical genus g* we mean the genus of the smooth Riemann surface obtained from the curve C by its completion with the points at infinity and the *normalization* of the obtained compact curve $\overline{C}$ (we separate the intersecting local branches and smooth the singular points).

Although the subject seems to be absolutely classical it turns out that there are surprisingly few results in this area. In particular it not much known about the classification of plane affine curves.

First essential result is the following *Abhyankar-Moh-Suzuki theorem* [1, 12] (see also [15, 6]):

If the curve C is smooth and diffeomorphic with $\mathbb{C}$ *then it can be straightened to the line y = 0 by means of a composition of* elementary automorphisms $\mathbb{C}^2$ *of the form*

$$(x,y)\to(x+P(y),y) \text{ or } (x,y)\to(x,y+Q(x))$$

This result has applications in the famous Jacobian Conjecture. I recall that the problem deals with polynomial maps

$$H:(x,y)\to(f(x,y),g(x,y))$$

with constant nonzero Jacobian

$$\det\begin{pmatrix}\partial f/\partial x & \partial f/\partial y\\ \partial g/\partial x & \partial g/\partial y\end{pmatrix}=1$$

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The Jacobian Conjecture [16] states that in this case the map H is invertible.

It turns out that one can characterize those transformations of the plane which are invertible; (of course, this does not solve the Jacobian Conjecture). This is done in the following *Jung-van der Kulk theorem* [13]:

Any automorphism of the complex plane is a composition of elementary automorphisms and a linear transformation.

It follows directly from the AMS theorem. Indeed, since the map H is a holomorphic diffeomorphism, the curve $g(x; y) = 0$ is diffeomorphic with the complex line. Applying the composition of elementary transformations from the AMS theorem and scaling the variable y, we reduce the map H to

$$(f(x,y),y)$$

Here the Jacobian condition reads as $\frac{\partial f}{\partial x} \equiv 1$, thus $f = x + P(y)$.

The curves diffeomorphic with the complex line a contractible; their Betti numbers are: $b^0 = 1$, $b^1 = 0$, $b^2 = 0$. (In what follows we shall consider only irreducible curves with $b^0 = 1$ and $b^2 = 0$). But there exist also other curves with $b^1 = 0$. These are the *quasi-homogeneous curves* of the form

$$x^k = y^l$$

where k a l are relatively prime. Of course, these curves have singularity at $x = y = 0$. The following result belongs to Lin and Zaidenberg [14]:

Any singular contractible algebraic curve in $\mathbb{C}^2$ is equivalent (i.e. via an automorphism of the plane) to some quasi-homogeneous curve.

1. Curves with $b^1 = 1$

Together with Maciej Borodzik from University of Warsaw I started to study the problem of classification of curves with $b^1 = 1$. From the topological point of view such a curve is of one of the two types:

(a) the curve is rational (i.e. with genus $g = 0$) with one point at infinity and one self-intersection (called *parametric lines*),

(b) the curve is rational with two points at infinity and without finite self-intersections (called *annuli*).

Of course, these curves can have other singular points, but these singularities must be *cuspidal*, i.e. the curve has locally one branch (like $x^2 = y^3$).

We found 44 essentially different cases of such curves; their list is given below.

These curves are given in parametric form: $x=\varphi(t)$, $y=\psi(t)$ where φ, ψ are polynomials in the parametric lines case and Laurent polynomials in the annuli case. The natural numbers $\alpha,\beta,\gamma,\delta$ appearing below satisfy the relation $\alpha\delta-\beta\gamma=1$.

Parametric lines [3]:

(a) $x=t^2,\quad y=t^{2l+1}(t^2-1)^k,\quad k=1,2,....,\quad l=0,1,....;$

(b) $x=t^3,\quad y=t^{3k+1}(t-1),\quad k=1,2,....;$

(c) $x=t^4,\quad y=t^{4k+1}(t-1),\quad k=1,2,....;$

(d) $x=t^4,\quad y=t^{4k+2}(t-1),\quad k=0,1,....;$

(e) $x=t^6,\quad y=t^{6k+2}(t-1),\quad k=1,2,.....;$

(f) $x=t^6,\quad y=t^{6k+3}(t-1),\quad k=0,1,.....;$

(g) $x=t^{\alpha}(t-1)^{k\beta},\quad y=t^{\gamma}(t-1)^{k\delta},\quad k=1,2,....,\quad 2<\alpha+k\beta<\gamma+k\delta;$

(h) $x=t^{2\alpha}(t-1)^{2\beta},\quad y=t^{2\gamma}(t-1)^{2\delta},\quad \alpha+\beta<\gamma+\delta;$

(i) $x=t^{\alpha}(t-1)^{k\beta-\alpha},\quad y=t^{\gamma}(t-1)^{k\delta-\gamma},\quad k=1,2,....,\quad 2<k\beta<k\delta;$

(j) $x=t^2(t-1),\quad y=x^k\cdot t(t-\frac{4}{3}),\quad k=1,2,....;$

(k) $x=t^3(t-1),\quad y=x^k\cdot t(t-\frac{3}{2}),\quad k=1,2,....;$

(l) $x=[t(t-1)]^{2k},\quad y=x^l\cdot[t(t-1)]^k(t-\frac{1}{2}),\quad k=1,2,....,\quad l=0,1,....;$

(m) $x=[t(t-1)]^{2k+1},\quad y=x^l\cdot[t(t-1)]^k(t-\frac{1}{2}),\quad k=0,1,....,\quad l=1,2,....;$

(n) $x=[t^k(t-1)^{k+1}]^2,\quad y=x^l\cdot t^k(t-1)^{k+1}(t-\frac{1}{2}),\quad k=1,2,....,\quad l=0,1,....;$

(o) $x=t^{2k+1}(t-1)^{2k+3},\quad y=x^l\cdot t^k(t-1)^{k+1}(t-\frac{1}{2}),\quad k=0,1,....,\quad l=1,2,....;$

(p) $x=[t(t-1)]^3,\quad y=x^k\cdot t(t-1)(t-\frac{1}{2}-\frac{1}{6}\sqrt{-3}),\quad k=1,2,.....;$

(q) $x=t^3-3t,\quad y=t^4-2t^2$

(r) $x=t^3-3t,\quad y=t^5-2\sqrt{-2}t^4+11\sqrt{-2}t^2-\frac{37}{4}t;$

(s) $x=t^3-3t,\quad y=t^5+10t^4+80t^2-205t;$

(t) $x=t^3-3t,\quad y=t^5-\frac{5}{2}t^4+5t^2-5t;$

(u) $x=t^3-3t,\quad y=t^5-\frac{7}{2}t^4-43t^2+11t;$

Annuli [4]:

(a) $x = t^m$, $y = t^n + b_1 t^{-m} + b_2 t^{-2m} + \ldots + b_k t^{-km}$, $\gcd(m,n) = 1$,
$k = 0,1,\ldots$, $b_j \in \mathbb{C}$, $(b_k = 1$ *when* $k > 0)$;

(b) $x = t(t-1)$, $y = R_{k,m}(\frac{1}{t})$, $k = 1,2,\ldots$, $m = 0,1,\ldots$, $(k,m) \neq (1,0),(2,0),(1,1)$,

and $R_{k,m}$ are Laurent polynomials defined by

$$R_{0,m}(u) = (\frac{1}{u} - \frac{1}{2})^{2m+1},\ R_{k+1,m}(u) = [R_{k,m}(u) - R_{k,m}(1)]\frac{u^2}{u-1};$$

(c) $x = t^{mn}(t-1)$, $y = S_k(\frac{1}{t})$, $k = 1,2,\ldots$, $n = 2,3,\ldots$, $mn \geq 2$, and S_k are defined

by: $S_0(u) = u^n$, $S_{k+1}(u) = [S_k(u) - S_k(1)]\frac{u^{mn+1}}{u-1}$;

(d) $x = t^{mn-1}(t-1)$, $y = T_k(\frac{1}{t})$, $k = 1,2,\ldots$, $n = 2,3,\ldots$, $mn \geq 3$, and T_k are defined

by: $T_0(u) = u^n$, $T_{k+1}(u) = [T_k(u) - T_k(1)]\frac{u^{mn}}{u-1}$;

(e) $x = t^{mn}(t-1)$, $y = U_k(\frac{1}{t})$, $k = 1,2,\ldots$, $n = 2,3,\ldots$, $mn \geq 2$,

and $U_0(u) = u^{-n}$, $U_{k+1}(u) = [U_k(u) - U_k(1)]\frac{u^{mn+1}}{u-1}$;

(f) $x = t^{mn-1}(t-1)$, $y = V_k(\frac{1}{t})$, $k = 1,2,\ldots$, $n = 2,3,\ldots$, $mn \geq 4$,

and $V_0(u) = u^{-n}$, $V_{k+1}(u) = [V_k(u) - V_k(1)]\frac{u^{mn}}{u-1}$;

(g) $x = t^2(t-1)$, $y = W_k(\frac{1}{t})$, $k = 1,2,\ldots$,

and $W_1(u) = 3u - u^2$, $W_{k+1}(u) = [W_k(u) - W_k(1)]\frac{u^3}{u-1}$;

(h) $x = t^3(t-1)$, $y = X_k(\frac{1}{t})$, $k = 1,2,\ldots$,

and $X_1(u) = 2u^2 - u^3$, $X_{k+1}(u) = [X_k(u) - X_k(1)]\frac{u^4}{u-1}$;

(i) $x = t^3(t-1)$, $y = Y_k(\frac{1}{t})$, $k = 1,2,\ldots$,

and $Y_1(u) = 2u^2 + u^3$, $Y_{k+1}(u) = [Y_k(u) - Y_k(1)]\frac{u^4}{u-1}$;

(j) $x = Z_{m,n}(t)$, $y = t + \frac{1}{t}$, $0 \leq m \leq n$, $(m,n) \neq (0,0)$ and the polynomials $Z_{m,n}$

are defined by $Z_{m,n}(t)-Z_{m,n}(\frac{1}{t})=(t-1)^{2m+1}(t+1)^{2n+1}t^{-m-n-1}$;

(k) $x=(t-1)^3t^{-2}$, $y=x^k\cdot(t-1)(t-4)t^{-1}$, $k=1,2,...,$;

(l) $x=(t-1)^m t^{-pn}$, $y=(t-1)^k t^{-pl}$, ml - $nk=1, p=1,2,...,$;

(m) $x=(t-1)^{pm}t^{-n}$, $y=(t-1)^{pk}t^{-l}$, ml - $nk=1, p=2,3,...,$;

(n) $x=(t-1)^{2m}t^{-2n}$, $y=(t-1)^{2k}t^{-2l}$, ml - $nk=1$;

(o) $x=y^n\cdot(t-1)^{2m}(t+1)t^{-m}$, $y=(t-1)^{4m}t^{1-2m}$, $m=1,2,...,$ $n=0,1,...$;

(p) $x=(t-1)^4t^{-3}$, $y=x^k\cdot(t-1)^2(t-3)t^{-2}$, $k=0,1,...,$;

(q) $x=y^n\cdot(t-1)^{2m-1}(t+1)t^{-m}$, $y=(t-1)^{4m-2}t^{1-2m}$, $m=2,3,...,$ $n=0,1,...$;

(r) $x=y^n\cdot(t-1)^3(t+e^{\frac{\pi i}{3}})t^{-2}$, $y=(t-1)^6t^{-3}$, $n=0,1,...$;

(s) $x=t^{2n}(t^2+\sqrt{2}t+1)$, $y=t^{-2n-4}(t^2-\sqrt{2}t+1)$, $n=1,2,...$;

(t) $x=(t^2+t+\frac{2}{3})t^4$, $y=(t^2-t+\frac{1}{3})t^{-8}$;

(u) $x=(t-1)^2(t+2)t^{-1}$, $y=(t-1)^4(t+\frac{1}{2})t^{-2}$;

(v) $x=(t-1)^2(t+4+2\sqrt{5})t^{-1}$, $y=(t-1)^4(t+\frac{1}{4}(11+5\sqrt{5}))t^{-2}$;

(w) $x=(t-1)^2(t+2)t^{-1}$, $y=(t-1)^2(t+\frac{1}{2})t^{-2}$.

It looks rather complicated. Therefore I propose to look closely at the first positions from these two lists. For example, it is easy to check that the curve $x=t^2, y=(t^2-1)^k t^{2l+1}$ has only one self-intersection point $x=1$, $y=0$ corresponding to $t=\pm1$. Also it is clear that the curve $x=t^m$, $y=t^n+\beta_1 t^{-m}+...+\beta_k t^{-mk}$ for $m>0$ and gcd($m; n$) = 1 has two points at infinity and no self-intersections.

Let me present our approach to study complex curves. It relies upon investigation of some invariants of singularities and relations between them.

One of such invariants is the *number of double points of the singularity* denoted by δ (see [11]). It is the number of double points in a neighborhood of the singular point of a special generic perturbation of the curve. For instance, the cusp $x=t^2$, $y=t^3$ can be perturbed to the curve $x=t^2$, $y=t^3-\varepsilon t$ which has one self-intersection point; so $\delta=1$ for the cusp.

The rigorous definition uses the Hamiltonian vector field

$$X_f=\frac{\partial f}{\partial y}\frac{\partial}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial}{\partial y}$$

tangent to the curve $C=\{f=0\}$. I recall that the normalization of the curve at singular point (which we assume equal 0) is the mapping N from disjoint union of smooth discs $A_j, j=1,....,r$, such that each map $N|_{A_j}$ defines a parametrization of the local branch $(C_j,0)$ of the germ $(C,0)$. Let $p_j\in A_j$ be the preimages of the singular point 0. We put

$$\delta=\sum_{j=1}^{r} i_{p_j}Y$$

where $Y=N^*X_f$ is the pull-back of X_f to $\bigcup A_j$ and $i_{p_j}Y$ denotes the index of the vector field at singular point. In the case of cuspidal singularity (r = 1) we have the formula

$$\delta=\frac{1}{2}\mu$$

where μ is the *Milnor number* of the singularity.

Another invariant of the singularity is its *codimension* denoted by ν. It is the codimension of the stratum (in the space of all germs) defining the topological type of the singularity. In the case of cuspidal singularity we can represent the curve in the form of the following *Puiseux expansion*

$$x=\tau^n,\ \ y=c_1\tau+c_2\tau^2+\ldots,$$

Then ν is the number of vanishing `essential' Puiseux coefficients c_i. For example, if n = 2 then the essential coefficients are $c_{1,}c_3,c_5,...$

In the paper [3] we proved the following inequality

$$\mu\le n\cdot\nu \tag{2.1}$$

In the case of multibranched singularity the codimension ν is defined similarly and an inequality similar to the above holds true.

A polynomial curve $x=\varphi(t),\ y=\psi(t)$ where φ and ψ are generic polynomials of degree p and q has

$$\delta_{\max}=\frac{1}{2}[(p-1)(q-1)-(p'-1)],\quad p'=\gcd(p,q) \tag{2.2}$$

simple double points. When the curve has simple topology (few self-intersections) the double points become hidden at cuspidal singular points and at infinity. We have then the identity

$$\sum \delta_{t_j} = \delta_{\max} \tag{2.3}$$

where δ_{t_j} is the number of double points hidden in the singular points t_j (also at $t=\infty$).

We have the simple inequality

$$\sum (n_{t_j} - 1) \le p - 1$$

which follows from differentiation of $\varphi(t)$. If we were able to estimate $\sum \nu_{t_j}$, i.e. the sum of codimensions of singularities, then formulas (2.1), (2.2) and (2.3) would reduce the range of possible curves.

Above lists of curves were detected under conjectured assumption that $\sum \nu_{t_j} \le$ dim(space of curves modulo equivalences).

We have not succeeded to prove this *codimension conjecture*. In [5] we obtained rather strong bounds (yet not optimal) using so-called *Bogomolov-Miyaoka-Yau inequality* (see [10]). In the case of smooth compact complex surface of general type it is the inequality

$$c_1^2 \le 3c_2$$

for the Chern classes of the surface. In our situation it takes much more omplicated form and I do not want to go into details.

3. Other applications

When we talked about our classification in a conference at Oberwolfach V. Lin became interested in our results and asked us whether our methods could be applied to estimation of the number of singular points of annuli. He told us about the conjecture

$$\#(\text{singular points}) \le 2b^1 + 1$$

which sometimes is called the *Lin-Zaidenberg conjecture*.

It turned out that our methods are strong enough and the bound

$$\#(\text{singular points for annulus}) \le 3$$

was proved in [9].

Borodzik continued investigation of the number of singular points individually. In [2] he considered curves of arbitrary genus g with one point at infinity (thus $b^1 = 2g$). He obtained the following bound

$$\#(\text{singular points}) \leq \frac{17}{11} b^1$$

for large g.

Other application of our methods is related with the *16^th^ Hilbert's problem*, i.e. the problem of estimation of the number of *limit cycles* (isolated periodic solutions) of polynomial vector fields. In general it is a very difficult problem.

We considered the *Lienard system*

$$\dot{x} = y + F(x), \quad \dot{y} = -G'(x) \tag{3.1}$$

where

$$F(x) = a_2 x^2 + \ldots + a_{m+1} x^{m+1}, \qquad G(x) = b_2 x^2 + \ldots + b_{n+1} x^{n+1}$$

and we can assume $b_2 = 1$. So the point $x = y = 0$ is singular for (3.1). We ask about the number of small limit cycles, i.e. limit cycles which bifurcate from the singular point with variation of the parameters a_i and b_j of the system. It is so-called *generalized Hopf bifurcation*.

Note that in the case when $F = \tilde{F} \circ \omega(x)$ and $G = \tilde{G} \circ \omega(x)$, $\omega(x) = x^2 + \ldots$, the system has *center* (locally all phase curves are closed). This follows from the fact that in system (3.1) we can make the substitution $x' = \omega(x)$ and the phase portrait of system (3.1) arises as the pull-back of the phase portrait of corresponding system in the (x', y) - plane. In such case we say that the system is *invertible*.
In order to find the *maximal cyclicity H* we should consider the Poincare *return map* $P: S \to S$, where the segment $S = \{(r,0) : r \geq 0\}$ is a section, defined by means of the trajectories of system (3.1) which start from S. The map P is real analytic and its Taylor expansion is following

$$P(r) = r + L_1 r(1 + \ldots.) + L_3 r^3 (1 + \ldots) + \ldots$$

where L_j are so-called *Poincare-Lyapunov focus numbers*. Let the *cyclicity* ν be defined such that $L_1 = L_3 = \ldots = L_{2\nu-1} = 0$ but $L_{2\nu+1} \neq 0$ the non-invertible case; otherwise we put $\nu = \infty$. Then, be definition

$$H = \max\{\nu : \nu < \infty\}$$

With system (3.1) we associate the rational curve

$$C : x \to (X, Y) = (F(x), G(x))$$

with singularity at $X = Y = 0$. It admits the Puiseux expansion near this singularity

$$X = c_2 Y + c_3 Y^{\frac{3}{2}} + c_4 Y^2 + \ldots$$

where the Puiseux coeffcients turn out to be polynomials of $a = (a_2, \ldots., a_{m+1})$ and $b = (1, b_3, \ldots., b_{n+1})$. It turns out that

$$L_{2j-1} = c_{2j-1}$$

i.e. the essential Puiseux coeficients of the curve *C* coincide with the focus numbers and the maximal cyclicity of the focus of the Lienard system coincides with the maximal codimension of the corresponding algebraic curve.

In the paper [8] we obtained the following estimates for the maximal cyclicity:

$$H \le \delta_{\max} - 1$$

where $\delta_{\max}$ is defined in (2.2) with $p = m + 1nq - n + 1$, and

$$H \le \frac{1}{4}(mn + 3m + 3n + 1)$$

4. Geometry of Puiseux expansions

Consider the space *Puis* $= \{\xi = (\varphi, \psi)\}$ of polynomial curves $x = \varphi(t),\ y = \psi(t)$ where

$$\varphi = t^p + a_1 t^{p-1} + \ldots + a_p, \quad \psi = t^q + b_1 t^{q-1} + \ldots + b_q$$

and $a_1 = 0$ (we can achive it by scaling *x, y, t* and shifting *t*). This space is isomorphic with $\mathbb{C}^{p+q-1}$.

We shall consider Puiseux expansions of curves from *Curv* at infinity. To this aim we write $t = x^{\frac{1}{p}}(1 + \ldots) \to \infty$ and we substitute it into the equation $y = \psi(t)$. We obtain

$$y = x^{\frac{q}{p}} + c_1 x^{\frac{q-1}{p}} + \ldots.,$$

where c_j are polynomial functions on the space *Curv*.

We define the space of truncated Puiseux series at infinity

$$Puis = \{x^{\frac{q}{p}} + c_1 x^{\frac{q-1}{p}} + \ldots + c_{p+q-1} x^{\frac{1-p}{p}}\}$$

which is also diffeomorphic with $\mathbb{C}^{p+q-1}$. We have a natural polynomial map

$$Expan : Curv \to Puis$$

It turns out that this map has a series of unexpected interesting properties (see [7]).

The first relates singularities of the map *Expan* with singularities of suitable curves in *Curv*:

A parametric curve $\xi_0 = (\varphi_0, \psi_0)$ *represents a critical point of the map Expan if and only if it is singular in the sense that* $\frac{d}{dt}\xi(t_0) = 0$ *for some* t_0.

In [7] the reader will find additional informations about relations between the types of singularities of the map *Expan* at ξ_0 and singularities of ξ_0 at t_0.

Let me recall that a continuous map F between non-compact metric spaces X and Y is *proper* when the preimage of any compact subset of Y is compact. When F is nonproper its *nonproperness set* $S(F)$ is defined as the set of those points $y_0 \in Y$ that there exists a sequence of points $y_n \to y_0$ and a sequence of preimages of these points $x_n \in F^{-1}(y_n)$ which is divergent in X.

Let $p' = \gcd(p, q)$ and $p_1 = \frac{p}{p'}$, $q_1 = \frac{q}{p'}$. It turns out that:

if $p' = 1$ *then the map Expan is a proper ramified covering of degree*

$$\frac{(p+q-1)!}{p!q!}$$

and if $p' > 1$ *then the map Expan is nonproper and*

$$S(Expan) = \{c_1 = 0\}$$

We failed in calculation of the topological degree of the map *Expan* in the general nonproper case. But we proved that:

if $p' = 2$ *then this degree equals*

$$\frac{(p+q-1)!}{p!q!} - \frac{1}{2}\left(\frac{(p_1+q_1-1)!}{p_1!q_1!}\right)^2$$

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