SOLUTION OF 2D HYPERBOLIC EQUATION BY MEANS OF THE BEM USING DISCRETIZATION IN TIME

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Abstract. The hyperbolic equation (2D problem) supplemented by adequate boundary and initial conditions is considered. To solve the problem the boundary element method using discretization in time is adapted. In the final part of the paper the example of computations is shown.

Introduction

The following hyperbolic equation is considered

$$(x_1, x_2) \in \Omega: \quad \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} = \nabla^2 \theta \tag{1}$$

where $\theta = \theta(x_1, x_2)$ is an unknown function, $\{x_1, x_2\}$ are the spatial co-ordinates and t is the time.

$$\begin{aligned} (x_1, x_2) \in \Gamma_1 &: \quad \theta = \theta_b \\ (x_1, x_2) \in \Gamma_2 &: \quad q = -\frac{\partial \theta}{\partial n} = q_b \end{aligned} \tag{2}$$

The equation (1) is supplemented by boundary conditions and initial ones

$$\begin{aligned} t = 0 &: \quad \theta = \theta_0 \\ &\left.\frac{\partial \theta}{\partial t}\right|_{t=0} = 0 \end{aligned} \tag{3}$$

where θ_b, q_b, θ_0 are the known functions.

The boundary element method using discretization in time [1-3] is proposed for such problem solution. The algorithm bases on the weighted residual formulation. Numerical model for constant boundary elements and constant internal cells is presented. In the final part the results of computations are shown.

1. Boundary integral equation

To solve the problem (1), the BEM using discretization in time is applied [1, 2]. So, the time grid

$$0 = t^0 < t^1 < \ldots < t^{f-2} < t^{f-1} < t^f < \ldots < t^F < \infty \tag{4}$$

with constant step $\Delta t = t^f - t^{f-1}$ is introduced.
For time interval $\left[t^{f-2}, t^f \right]$ the following approximation of hyperbolic equation can be taken into account

$$\frac{\theta^f - 2\theta^{f-1} + \theta^{f-2}}{(\Delta t)^2} + \frac{\theta^f - \theta^{f-1}}{\Delta t} = \nabla^2 \theta^f \tag{5}$$

or

$$\nabla^2 \theta^f - \left(\beta^2 + \beta\right)\theta^f + \left(2\beta^2 + \beta\right)\theta^{f-1} - \beta^2 \theta^{f-2} = 0 \tag{6}$$

where $\beta = 1/\Delta t$.
The equation (5) can be written in the form

$$\nabla^2 \theta^f - A\theta^f + B\theta^{f-1} - C\theta^{f-2} = 0 \tag{7}$$

where

$$A = \beta^2 + \beta, \;\; B = 2\beta^2 + \beta, \;\; C = \beta^2 \tag{8}$$

For equation (7) the weighted residual criterion is applied [1]

$$\int_\Omega \left(\nabla^2 \theta^f - A\theta^f + B\theta^{f-1} - C\theta^{f-2} \right) \theta^*(\xi, x)\, d\Omega = 0 \tag{9}$$

where $\xi \in (0,1)$ is the observation point, $\theta^*(\xi, x)$ is the fundamental solution and this function should fulfil the equation

$$\nabla^2 \theta^*(\xi, x) - A\theta^*(\xi, x) = -\delta(\xi, x) \tag{10}$$

where $\delta(\xi, x)$ is the Dirac function. It can be check that the following function

$$\theta^* = \frac{1}{2\pi} K_0\left(r\sqrt{A} \right) \tag{11}$$

fulfills the equation (10). In equation (11) r denotes the distance between the points ξ and x.

Additionally, the function $q^{*}(\xi, x)$ resulting from fundamental solution is defined

$$q^{*}(\xi,x)=-\frac{\partial\theta^{*}(\xi,x)}{\partial n} \tag{12}$$

and it can be calculated analytically

$$q^{*}=\frac{d\sqrt{A}}{2\pi r}K_1\left(r\sqrt{A}\right) \tag{13}$$

where

$$d=(x_1-\xi_1)\cos\alpha_1+(x_2-\xi_2)\cos\alpha_2 \tag{14}$$

while $\mathbf{n}=[\cos\alpha_1,\cos\alpha_2]$.
In equations (11), (13) the functions $K_0(\cdot)$ and $K_1(\cdot)$ are the modified Bessel's functions of first kind, zero and first order, respectively.
Applying to the first component of equation (9) the 2nd Green formula one obtains

$$\int_\Omega\left(\nabla^2\theta^f\right)\theta^{*}(\xi,x)d\Omega=\int_\Omega\left[\nabla^2\theta^{*}(\xi,x)\right]\theta^f d\Omega+ \int_\Gamma\left[\theta^{*}(\xi,x)\frac{\partial\theta^f}{\partial n}-\theta^f\,\frac{\partial\theta^{*}(\xi,x)}{\partial n}\right]d\Gamma \tag{15}$$

this means

$$\int_\Omega\left(\nabla^2\theta^f\right)\theta^{*}(\xi,x)d\Omega=\int_\Omega\left[\nabla^2\theta^{*}(\xi,x)\right]\theta^f d\Omega+ \int_\Gamma\left[q^{*}(\xi,x)\theta^f-\theta^{*}(\xi,x)q^f\right]d\Gamma \tag{16}$$

where $q^f=-\partial\theta^f/\partial n$.
Putting (16) into (9) one has

$$\int_\Omega\left[\nabla^2\theta^{*}(\xi,x)-A\theta^{*}(\xi,x)\right]\theta^f d\Omega+\int_\Gamma\left[q^{*}(\xi,x)\theta^f-\theta^{*}(\xi,x)q^f\right]d\Gamma+ \int_\Omega\left[B\theta^{f-1}-C\theta^{f-2}\right]\theta^{*}(\xi,x)d\Omega \tag{17}$$

Taking into account the property (10) one obtains

$$\theta\left(\xi,t^f\right)+\int_\Gamma\theta^{*}(\xi,x)q^f d\Gamma=\int_\Gamma q^{*}(\xi,x)\theta^f d\Gamma+\int_\Omega\left[B\theta^{f-1}-C\theta^{f-2}\right]\theta^{*}(\xi,x)d\Omega \tag{18}$$

For $\xi \to \Gamma$ the boundary integral equation is as follows

$$B(\xi)\theta\left(\xi,t^f\right)+\int_\Gamma \theta^*(\xi,x)q^f d\,\Gamma=\int_\Gamma q^*(\xi,x)\theta^f d\,\Gamma+ \int_\Omega\left[B\theta^{f-1}-C\theta^{f-2}\right]\theta^*(\xi,x)d\Omega \tag{19}$$

where $B(\xi)\in(0,1)$ is the coefficient connected with the position of the point ξ on the boundary Γ.

2. Numerical model

To solve the boundary integral equation (19) the boundary Γ is divided into N constant elements and the interior Ω is divided into L constant internal cells. The approximate form of equation (19) for boundary nodes i, $i=1,2,\dots,N$ is following

$$\frac{1}{2}\theta_i^{\,f}+\sum_{j=1}^{N}q_j^{\,f}\int_{\Gamma_j}\theta^*\left(\xi^i,x\right)d\,\Gamma_j=\sum_{j=1}^{N}\theta_j^{\,f}\int_{\Gamma_j}q^*\left(\xi^i,x\right)d\,\Gamma_j+ \sum_{l=1}^{L}\left(B\theta_l^{\,f-1}-C\theta_l^{\,f-2}\right)\int_{\Omega_l}\theta^*\left(\xi^i,x\right)d\Omega_l \tag{20}$$

or

$$\sum_{j=1}^{N}G_{i\,j}\,q_j^{\,f}=\sum_{j=1}^{N}H_{i\,j}\,\theta_j^{\,f}+\sum_{l=1}^{L}P_{il}\left(B\theta_l^{\,f-1}-C\theta_l^{\,f-2}\right) \tag{21}$$

where

$$G_{i\,j}=\int_{\Gamma_j}\theta^*\left(\xi^i,x\right)d\,\Gamma_j\,,\quad H_{i\,j}=\int_{\Gamma_j}q^*\left(\xi^i,x\right)d\,\Gamma_j \tag{22}$$

and

$$H_{i\,j}=\begin{cases}\displaystyle\int_{\Gamma_j}q^*\left(\xi^i,x\right)d\,\Gamma_j, & i\neq j\\ -0.5, & i=j\end{cases} \tag{23}$$

while

$$P_{i\,l}=\int_{\Omega_l}\theta^*\left(\xi^i,x\right)d\Omega_l \tag{24}$$

The system of equations (21) allows to determine the 'missing' boundary values of θ_j^f and q_j^f. Next, the values θ_i^f at the internal points are calculated by means of the formula

$$\theta_i^f = \sum_{j=1}^{N} H_{ij}\,\theta_j^f - \sum_{j=1}^{N} G_{ij}\,q_j^f + \sum_{l=1}^{L} P_{il}\left(B\theta_l^{f-1} - C\theta_l^{f-2}\right) \tag{25}$$

It should be pointed out that the integrals (22), (23), (24) are determined using the Gaussian integration method.

3. Results of computations

The square of dimensions 1×1 is considered. The boundary is divided into 40 constant boundary elements, the interior is divided into 100 constant internal cells. Time step equals $\Delta t = 0.05$. On the bottom and left surface the boundary condition $q_b = 0$ is assumed, on the upper and right surface the boundary condition $\theta_b = 1$ is accepted. The initial condition $\theta = 0$ is taken into account. In Figure 1 the distribution of function θ for times 0.2, 0.4, 0.6 and 0.8 is shown.

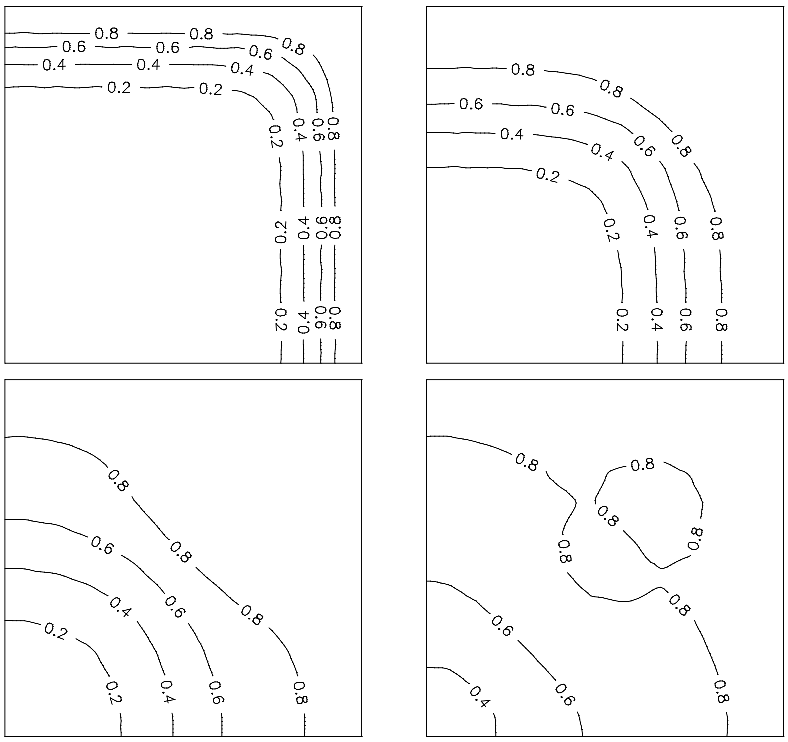

Fig. 1. Distribution of function θ for $t = 0.2$, $t = 0.4$, $t = 0.6$ and $t = 0.8$

Figure 2 illustrates the course of function θ at the central point of the domain considered.

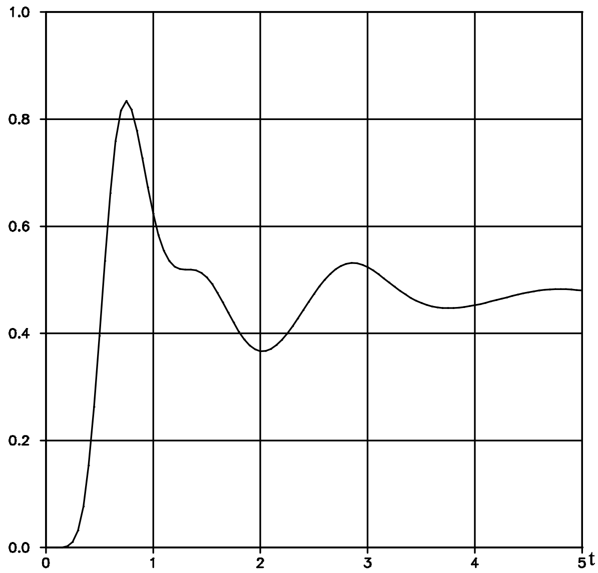

Fig. 2. Course of function θ at the central point

It should be pointed out that the method described in this paper constitutes the extended on 2D problem version of the algorithm presented in [4] concerning the solution of 1D hyperbolic equation by means of the boundary element method using discretization in time.

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