SOLUTION OF 2D HYPERBOLIC EQUATION BY MEANS OF THE BEM USING DISCRETIZATION IN TIME - ANALYSIS OF SOLUTION EXACTNESS

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Abstract. The hyperbolic equation (2D problem) supplemented by adequate boundary and initial conditions is considered. This equation is solved by means of the boundary element method using discretization in time. The aim of investigations is to analyze the influence of time step and the discretization assumed on the exactness of the obtained results.

Introduction

The following problem is considered

$$
\begin{array}{ll}
\Omega = \{ 0 < x_1 < 1 ,\ 0 < x_2 < 1 \} \ : & \dfrac{\partial^2 \theta}{\partial t^2} + \dfrac{\partial \theta}{\partial t} = \nabla^2 \theta \\
x_1 = 0 \ , \quad 0 < x_2 < 1 \ : & \theta = 1 \\
x_1 = 1 \ , \quad 0 < x_2 < 1 \ : & -\dfrac{\partial \theta}{\partial n} = 0 \\
x_2 = 0 \ , \quad x_2 = 1 \ , \quad 0 < x_2 < 1 \ : & -\dfrac{\partial \theta}{\partial n} = 0 \\
t = 0 \ : & \theta = 0 \\
t = 0 \ : & \dfrac{\partial \theta}{\partial t} = 0
\end{array}
\tag{1}
$$

where $\theta = \theta(x_1, x_2)$ is an unknown function, $\{x_1, x_2\}$ are the spatial co-ordinates and t is the time.

The aim of investigations is to solve the problem by means of the boundary element method using dicretization in time and to analyze the influence of time step and the discretization assumed on the results of computations.

1. Boundary element method using discretization in time

To solve the problem (1), the BEM using discretization in time is applied [1-4]. So, the time grid

$$0 = t^0 < t^1 < \ldots < t^{f-2} < t^{f-1} < t^f < \ldots < t^F < \infty \tag{2}$$

with constant step $\Delta t = t^f - t^{f-1}$ is introduced.
The boundary integral equation corresponding to the problem (1) has the following form [1]

$$\begin{gathered} B(\xi)\theta(\xi, t^f) + \int_\Gamma \theta^*(\xi, x) q^f \, d\Gamma = \int_\Gamma q^*(\xi, x)\theta^f \, d\Gamma + \\ \int_\Omega \left[B\theta^{f-1} - C\theta^{f-2}\right]\theta^*(\xi, x) d\Omega \end{gathered} \tag{3}$$

where $B(\xi) \in (0,1)$ is the coefficient connected with the position of the point ξ on the boundary Γ, $\theta^*(\xi, x)$ is the fundamental solution, $q^*(\xi, x) = -\partial\theta^*(\xi, x)/\partial n$ and $q^f = -\partial\theta^f/\partial n$.
For 2D problem the functions $\theta^*(\xi, x)$ and $q^*(\xi, x)$ are of the form [5]

$$\theta^* = \frac{1}{2\pi} K_0\left(r\sqrt{A}\right) \tag{4}$$

and

$$q^* = \frac{d\sqrt{A}}{2\pi r} K_1\left(r\sqrt{A}\right) \tag{5}$$

where $K_0(\cdot)$ and $K_1(\cdot)$ are the modified Bessel's function of first kind, zero and first order, respectively, r is the distance between the points ξ and x, while

$$d = (x_1 - \xi_1)\cos\alpha_1 + (x_2 - \xi_2)\cos\alpha_2 \tag{6}$$

where $\mathbf{n} = [\cos\alpha_1, \cos\alpha_2]$. In equations (3), (4), (5):

$$A = \beta^2 + \beta, \qquad B = 2\beta^2 + \beta, \qquad C = \beta^2 \tag{7}$$

where $\beta = 1/\Delta t$.
To solve the boundary integral equation (3) the boundary Γ is divided into N constant elements and the interior Ω is divided into L constant internal cells as shown in Figure 1.

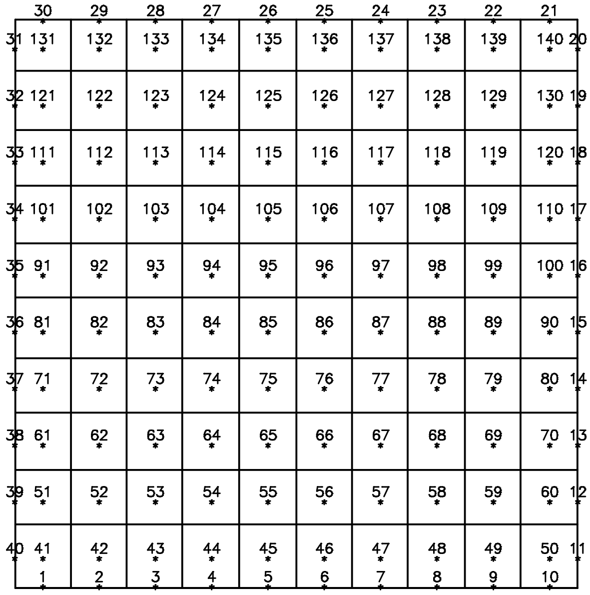

Fig. 1. Discretization

At the first step of computations the following system of algebraic equation should be solved ($i=1, 2, \ldots, N$)

$$\sum_{j=1}^{N} G_{ij}\, q_j^{\,f} = \sum_{j=1}^{N} H_{ij}\, \theta_j^{\,f} + \sum_{l=1}^{L} P_{il}\left(B\theta_l^{\,f-1} - C\theta_l^{\,f-2}\right) \tag{8}$$

The definitions of elements G_{ij}, H_{ij}, P_{il} are presented in [2].

At the second step of computations the internal values of function θ should be determined ($i = N+1, N+2, \ldots, N+L$)

$$\theta_i^{\,f} = \sum_{j=1}^{N} H_{ij}\, \theta_j^{\,f} - \sum_{j=1}^{N} G_{ij}\, q_j^{\,f} + \sum_{l=1}^{L} P_{il}\left(B\theta_l^{\,f-1} - C\theta_l^{\,f-2}\right) \tag{9}$$

2. Analysis of solution exactness

Application of the boundary element method using discretization in time requires the proper assumption of time step Δt and number of internal cells L. The testing computations presented in this paper have been done for time step $\Delta t = 0.05$, $0.1, 0.5$ and number of internal cells $L = 10\times 10$ (c.f. Figure 1), 20×20, 40×40. The successive variants of computations are collected in Table 1. To show convergence, the error of numerical solution is calculated

$$B^f = \frac{1}{L^2}\sum_{l=1}^{L}\left(\nabla_a{}^2\theta_l^f - A\theta_l^f + B\theta_l^{f-1} + C\theta_l^{f-2}\right)^2 \tag{10}$$

where $\nabla_a{}^2\theta_l^f$ denotes the approximation of operator $\nabla^2\theta$ at the internal point l.

Table 1. Variants of computations

Variant	Mesh	Time step Δt	Error E
1	10×10	0.05	0.321
2	10×10	0.10	0.094
3	10×10	0.50	0.017
4	20×20	0.05	0.099
5	20×20	0.10	0.036
6	20×20	0.50	0.011
7	40×40	0.05	0.037
8	40×40	0.10	0.018
9	40×40	0.50	0.008

In Figure 2 different locations of internal point l are shown (c.f. Figure 1) and for these cases the following approximations of operator $\nabla^2\theta$ have been taken into account

$$\begin{aligned}
&\text{a)}\ \nabla_a{}^2\theta_l^f = \frac{8\theta_1^f - 12\theta_l^f + 4\theta_2^f}{3h^2} + \frac{8\theta_3^f - 12\theta_l^f + 4\theta_4^f}{3h^2}\\
&\text{b)}\ \nabla_a{}^2\theta_l^f = \frac{\theta_1^f - 2\theta_l^f + \theta_2^f}{h^2} + \frac{8\theta_3^f - 12\theta_l^f + 4\theta_4^f}{3h^2}\\
&\text{c)}\ \nabla_a{}^2\theta_l^f = \frac{4\theta_1^f - 12\theta_l^f + 8\theta_2^f}{3h^2} + \frac{8\theta_3^f - 12\theta_l^f + 4\theta_4^f}{3h^2}\\
&\text{d)}\ \nabla_a{}^2\theta_l^f = \frac{8\theta_1^f - 12\theta_l^f + 4\theta_2^f}{3h^2} + \frac{\theta_3^f - 2\theta_l^f + \theta_4^f}{h^2}\\
&\text{e)}\ \nabla_a{}^2\theta_l^f = \frac{\theta_1^f - 2\theta_l^f + \theta_2^f}{h^2} + \frac{\theta_3^f - 2\theta_l^f + \theta_4^f}{h^2}\\
&\text{f)}\ \nabla_a{}^2\theta_l^f = \frac{4\theta_1^f - 12\theta_l^f + 8\theta_2^f}{3h^2} + \frac{\theta_3^f - 2\theta_l^f + \theta_4^f}{3h^2}
\end{aligned} \tag{11}$$

$$\text{g)}\ \nabla_a^{\,2}\theta_l^{\,f}=\frac{8\theta_1^{\,f}-12\theta_l^{\,f}+4\theta_2^{\,f}}{3h^2}+\frac{4\theta_3^{\,f}-12\theta_l^{\,f}+8\theta_4^{\,f}}{3h^2}$$

$$\text{h)}\ \nabla_a^{\,2}\theta_l^{\,f}=\frac{\theta_1^{\,f}-2\theta_l^{\,f}+\theta_2^{\,f}}{h^2}+\frac{4\theta_3^{\,f}-12\theta_l^{\,f}+8\theta_4^{\,f}}{3h^2}$$

$$\text{i)}\ \nabla_a^{\,2}\theta_l^{\,f}=\frac{4\theta_1^{\,f}-12\theta_l^{\,f}+8\theta_2^{\,f}}{3h^2}+\frac{4\theta_3^{\,f}-12\theta_l^{\,f}+8\theta_4^{\,f}}{3h^2}$$

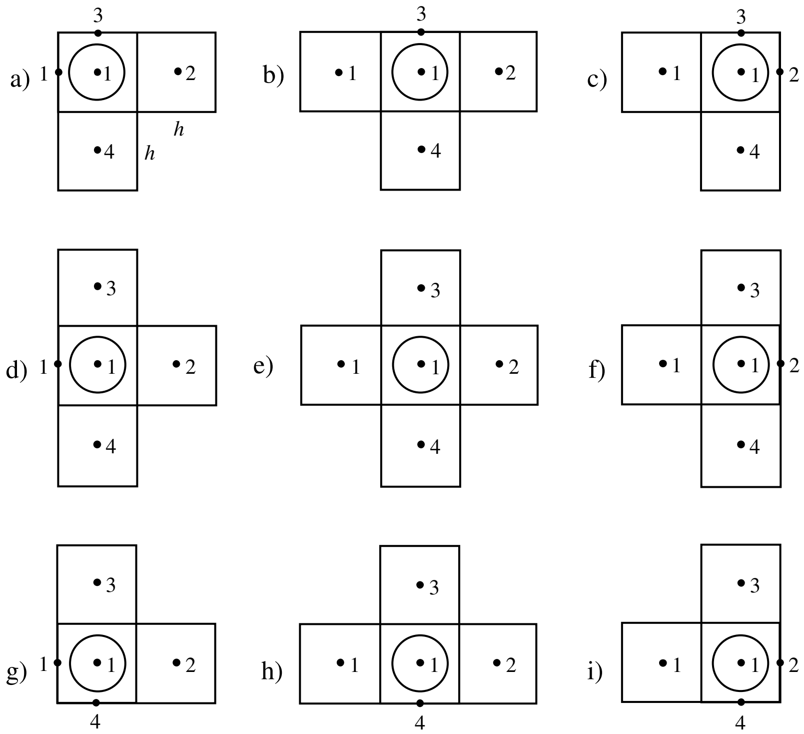

Fig. 2. Different locations of internal point l

In Figures 3-5 the course of function θ_l^f at the central point of domain considered for successive variants of computations is shown. Figure 6 illustrates the results for all variants. It is visible, that the results are strongly dependent on the time step assumed. In the last column of Table 1 the global error of numerical solution is presented, this means

$$E=\sqrt{\frac{1}{F}\sum_{f=1}^{F}B^f} \tag{12}$$

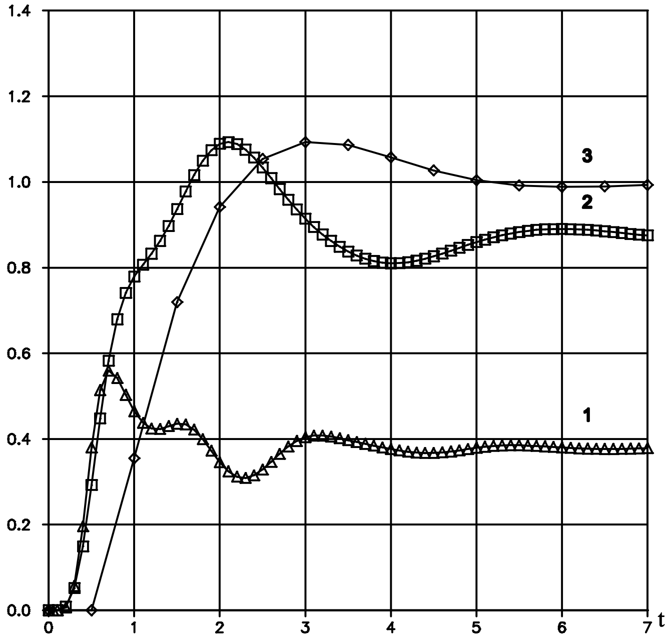

Fig. 3. Variants 1, 2, 3

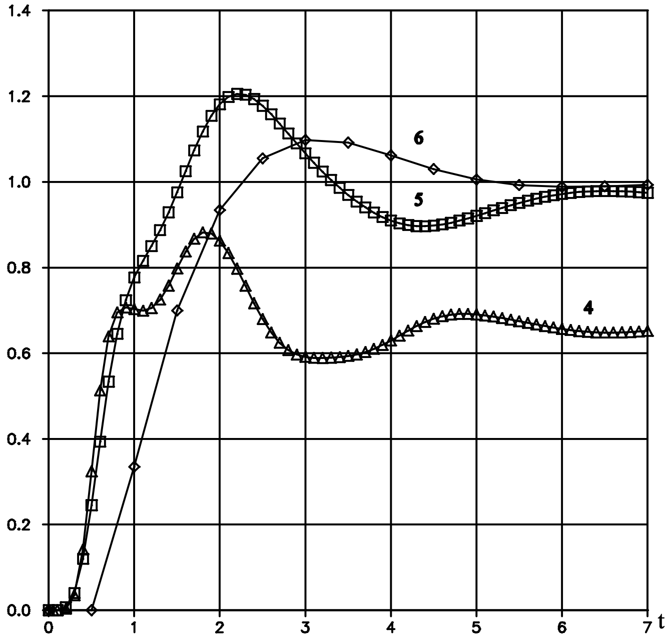

Fig. 4. Variants 4, 5, 6

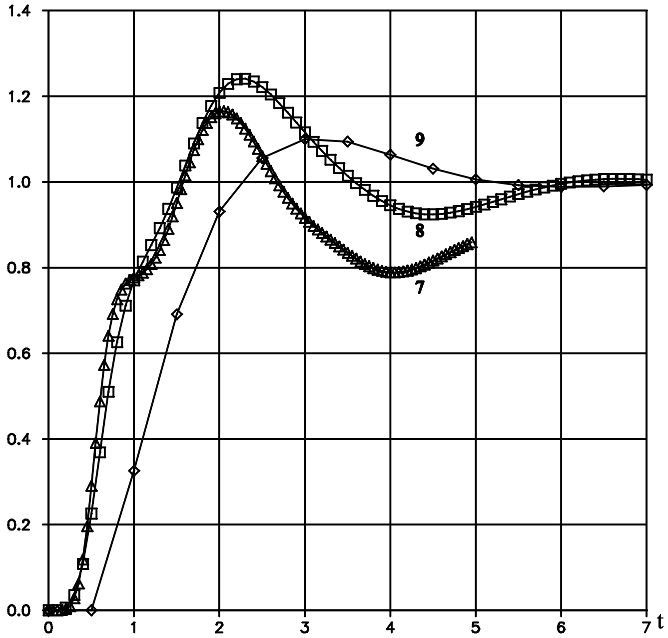

Fig. 5. Variants 7, 8, 9

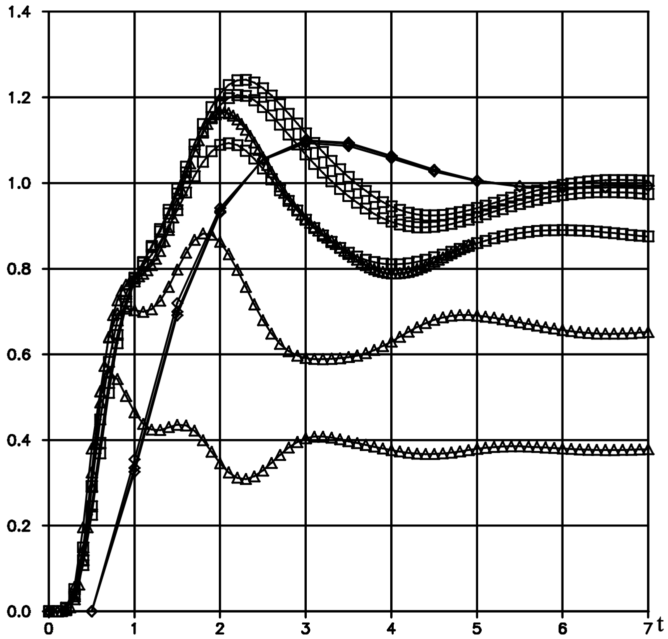

Fig. 6. All variants of computations

Conclusions

For variants 3, 6 and 9 corresponding to the small values of error (12) the results of computations are practically the same.

It should be pointed out that the computations have been done for the time steps greater than the optimum time steps resulting from calculations (see: Table 1) and for these values the errors were very small.

Summing up, the BEM using discretization in time constitutes the effective numerical method of hyperbolic equation solution under the assumption that the time step is proper. Additionally, if the small value of time step is needed then the big number L of internal cells should be taken into account.

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