EXPERIMENT DESIGN FOR PARAMETERS ESTIMATION OF NONLINEAR POISSON EQUATION - PART II

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Abstract. In this part of the paper the inverse problem consisting in the identification of unknown parameters p_1, p_2 appearing in conductivity $D(x) = p_1x_1x_2+p_2$ where $x = \{x_1, x_2\}$, $-1\le x_1, x_2\le 1$ is analyzed. To solve this problem the additional information connected with the knowledge of function $U(x)$ at the set of points (sensors) selected from the domain considered is necessary. The fundamental problem is the selection of sensors location and here the algorithm assuring the optimal sensors location is presented. In the final part of the paper the results of computations are shown.

1. Formulation of the problem

The following two-dimensional Poisson equation is considered [1]

$$
\begin{gathered}
(x_1, x_2) \in \Omega: \quad \frac{\partial}{\partial x_1}\left[(p_1 x_1 x_2 + p_2)\frac{\partial U(x_1, x_2)}{\partial x_1}\right] + \\
\frac{\partial}{\partial x_2}\left[(p_1 x_1 x_2 + p_2)\frac{\partial U(x_1, x_2)}{\partial x_2}\right] + Q(x_1, x_2) = 0
\end{gathered}
\tag{1}
$$

where $\Omega = \{x_1, x_2: -1\le x_1\le 1, -1\le x_2\le 1\}$, $Q(x_1, x_2)$ is the source function and p_1, p_2 are the unknown parameters.
The equation (1) is supplemented by boundary condition

$$
(x_1, x_2) \in \Gamma: \quad U(x_1, x_2) = 0 \tag{2}
$$

To identify the parameters p_1, p_2 the additional information connected with the knowledge of function $U(x)$ at the set of points (sensors) selected from the domain considered is necessary. The accuracy of identification depends significantly on the choice of sensors location and this problem is here discussed.

2. Algorithm of optimal sensors location

Let $X = \{x^1, x^2, \ldots, x^M\}$ denotes the set of spatial points at which measurements may be taken. The practical design problem consists in selection of corresponding weights $w_1, w_2, \ldots, w_M$ which define the best experimental conditions [1]. To solve this problem the following iterative algorithm under the assumption that number of unknown parameters equals 2 and number of sensors equals N can be applied. At first, the sensitivity matrix is constructed

$$\mathbf{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \\ \ldots & \ldots \\ Z_{M1} & Z_{M2} \end{bmatrix} \tag{3}$$

where

$$Z_{l1} = \frac{\partial U(x^l, p_1^0, p_2^0)}{\partial p_1^0}, \quad Z_{12} = \frac{\partial U(x^l, p_1^0, p_2^0)}{\partial p_2^0}, \quad l = 1,2,\ldots,M \tag{4}$$

and p_1^0, p_2^0 are the estimates of unknown parameters available e.g. from preliminary experiments.

Let

$$\mathbf{Z}(x^i) = [Z_{i1} \; Z_{i2}] \tag{5}$$

and

$$\mathbf{S}(x^i) = \mathbf{Z}^T(x^i)\,\mathbf{Z}(x^i) = \begin{bmatrix} Z_{i1}^2 & Z_{i1}Z_{i2} \\ Z_{i1}Z_{i2} & Z_{i2}^2 \end{bmatrix} \tag{6}$$

Step 1. Let $k = 0$ and *eps* is some positive tolerance. We assume that $S_0 = \{x^1, x^2, \ldots, x^N\}$ denotes the initial sub-set of X.

Step 2. We set

$$w_l^k = \begin{cases} \dfrac{1}{M} & \text{if } x^l \in S_k \\ 0 & \text{if } x^l \in X \setminus S_k \end{cases}, \quad l = 1,2,\ldots,M \tag{7}$$

Step 3. The following matrices are calculated

$$\mathbf{R}(\mathbf{w}^k) = \mathbf{Z}^T\,\mathbf{w}^k\,\mathbf{Z} \tag{8}$$

this means

$$\mathbf{R}(\mathbf{w}^k) = \begin{bmatrix} \sum_{l=1}^{M} w_l^k Z_{l1}^2 & \sum_{l=1}^{M} w_l^k Z_{l1} Z_{l2} \\ \sum_{l=1}^{M} w_l^k Z_{l1} Z_{l2} & \sum_{l=1}^{M} w_l^k Z_{l2}^2 \end{bmatrix} \tag{9}$$

and

$$\mathbf{P}(x^i, \mathbf{w}^k) = \mathbf{R}^{-1}(\mathbf{w}^k)\,\mathbf{S}(x^i) \tag{10}$$

where

$$\mathbf{w}^k = \begin{bmatrix} w_1^k & 0 & \dots & 0 \\ 0 & w_2^k & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & w_M^k \end{bmatrix} \tag{11}$$

Step 4. Find

$$x^{*k} = \min_{x^i \in S_k} [\mathit{trace}\,\mathbf{P}(x^i, w^k)] \tag{12}$$

and

$$x^{**k} = \max_{x^i \in X \setminus S_k} [\mathit{trace}\,\mathbf{P}(x^i, w^k)] \tag{13}$$

Step 5. If

$$\mathrm{trace}\,\mathbf{P}(x^{**k}, w^k) - \mathrm{trace}\,\mathbf{P}(x^{*k}, w^k) > \mathit{eps} \tag{14}$$

then set

$$S_{k+1} = \{S_k \setminus \{x^{*k}\}\} \cup \{x^{**k}\} \tag{15}$$

increase k by one and go to *Step 2*, otherwise *Stop.*

It should be pointed out that the sensitivity coefficients (4) are determined using the direct differentiation method presented in the first part of the paper [2].

3. Results of computations

The set of possible sensors location is determined by the following points

$$X = \left\{ x:\ x_{1i} = -1 + \frac{2}{9}(i-1),\, x_{2j} = -1 + \frac{2}{9}(j-1), \quad i, j = 1,2,\ldots,10 \right\} \tag{16}$$

The initial set of measurement points (20 sensors) is described as

$$S_0 = \left\{ x:\ x_{1i} = -1 + \frac{4}{9}(i-1),\ i = 1,2,\ldots,5,\ x_{2j} = -1 + \frac{6}{9}(j-1),\ j = 1,2,3,4 \right\} \tag{17}$$

In Figure 1 the initial location of sensors is marked by black circles, while Figure 2 illustrates the optimal sensors location obtained using the algorithm above presented.

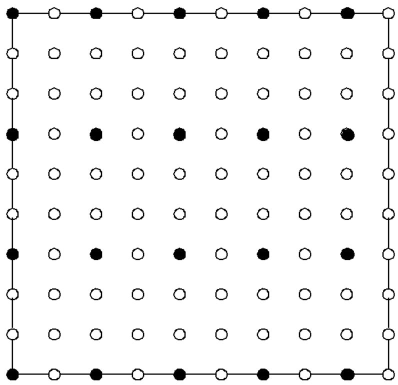

Fig. 1. Initial position of sensors

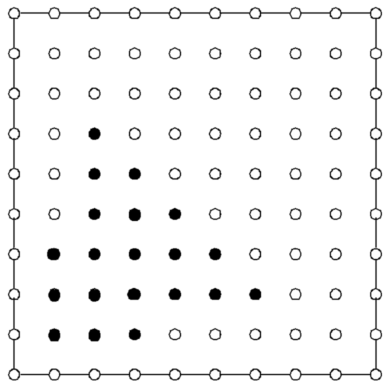

Fig. 2. Optimum position of sensors

References

[1] Patan M., Uciński D., Optimal location of sensors for parameter estimation of static distributed systems, R. Wyrzykowski et al. (eds.), PPAM 2001, LNCS 2328, 2002, 729-737.

[2] Majchrzak E., Freus K., Freus S., Experiment design for parameter estimation of nonlinear diffusion equation - Part I, Scientific Research of the Institute of Mathematics and Computer Science 2009, 1(8).