IDENTIFICATION OF INTERNAL HOLE PARAMETERS ON THE BASIS OF BOUNDARY TEMPERATURE

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Abstract. The Laplace equation describing temperature field in 2D domain with an internal hole of circle shape supplemented by adequate boundary conditions is considered. On the basis of known temperature at the fragment of boundary the position of circle center or its radius is identified. To solve the inverse problem discussed the least square criterion is formulated, and next the gradient method coupled with the boundary element method is applied. To determine sensitivity coefficients the shape sensitivity analysis is used. In the final part of the paper the examples of computations are shown.

1. Direct problem

The steady state temperature field $T(x, y)$ in domain Ω limited by boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ (Fig. 1) is described by the Laplace equation

$$(x, y) \in \Omega : \nabla^2 T(x, y) = 0 \tag{1}$$

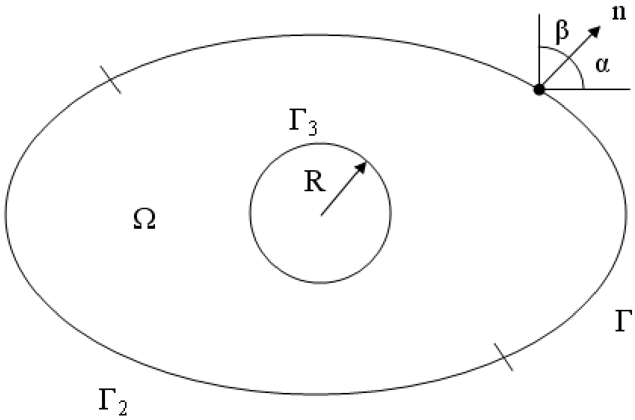

Fig. 1. Domain considered

The equation (1) is supplemented by the following boundary conditions

$$
\begin{aligned}
&(x, y) \in \Gamma_1: \quad q(x, y) = -\lambda\, n \cdot \nabla T(x, y) = q_b \\
&(x, y) \in \Gamma_2: \quad T(x, y) = T_{b1} \\
&(x, y) \in \Gamma_3: \quad T(x, y) = T_{b2}
\end{aligned} \tag{2}
$$

where T_{b1}, T_{b2}, q_b are known boundary temperatures and heat flux, respectively.

2. Boundary element method

The boundary integral equation corresponding to the equation (1) has the following form [1, 2]

$$
\begin{aligned}
&(\xi, \eta) \in \Gamma: \quad B(\xi, \eta) T(\xi, \eta) + \int_\Gamma q(x, y) T^*(\xi, \eta, x, y)\, \mathrm{d}\Gamma = \\
&\int_\Gamma T(x, y)\, q^*(\xi, \eta, x, y)\, \mathrm{d}\Gamma
\end{aligned} \tag{3}
$$

where (ξ, η) is the observation point, $q(x, y) = -\lambda\, \mathrm{n} \cdot \nabla T(x, y)$ is the boundary heat flux (λ is the thermal conductivity, $\mathrm{n} = [\cos\alpha \quad \cos\beta]$ is the unit outward vector normal to Γ - Figure 1), $B(\xi, \eta) \in (0, 1)$ is the coefficient connected with the local shape of boundary.

Function $T^*(\xi, \eta, x, y)$ is the fundamental solution and for the problem considered it has the following form

$$
T^*(\xi, \eta, x, y) = \frac{1}{2\pi\lambda} \ln \frac{1}{r} \tag{4}
$$

where r is the distance between the points (ξ, η) and (x, y).

The heat flux $q^*(\xi, \eta, x, y) = -\lambda\, \mathrm{n} \cdot \nabla T^*(\xi, \eta, x, y)$ resulting from the fundamental solution can be calculated in analytical way and then

$$
q^*(\xi, \eta, x, y) = \frac{d}{2\pi r^2} \tag{5}
$$

where

$$
d = (x - \xi) \cos\alpha + (y - \eta) \cos\beta \tag{6}
$$

To solve the equation (3) the boundary Γ is divided into N boundary elements Γ_j, $j = 1,2,..,N$ and the integrals appearing in equation (3) are substituted by the sums of integrals over these elements, namely

$$B(\xi_i, \eta_i)\, T(\xi_i, \eta_i) + \sum_{j=1}^{N} \int_{\Gamma_j} q(x, y)\, T^*(\xi_i, \eta_i, x, y)\, \mathrm{d}\Gamma_j = \sum_{j=1}^{N} \int_{\Gamma_j} T(x, y)\, q^*(\xi_i, \eta_i, x, y)\, \mathrm{d}\Gamma_j \tag{7}$$

where (ξ_i, η_i) denotes the boundary node.

In the case of linear boundary elements application one obtains the following system of equations ($i = 1,2,...,R$, R is the number of boundary nodes)

$$\sum_{r=1}^{R} G_{ir}\, q_r = \sum_{r=1}^{R} H_{ir}\, T_r \tag{8}$$

where $T_r = T(x_r, y_r)$, $q_r = q(x_r, y_r)$. The way of G_{ir} and H_{ir} computations is described in details in [2, 3].

The system of equations (8) can be written in the matrix form

$$\mathbf{G}\ \mathbf{q} = \mathbf{H}\ \mathbf{T} \tag{9}$$

It should be pointed out that in the system of equations (8) part of the boundary values (temperatures or heat fluxes) is known from the boundary conditions (2), while the remaining R boundary values (heat fluxes or temperatures) should be determined.

3. Shape sensitivity analysis - implicit approach

We assume that the shape parameter b corresponds to the radius R of internal hole or corresponds to the position of its centre, this means $b = x_s$ or $b = y_s$, where (x_s, y_s) is the centre of circle - Figure 1. The implicit differentiation method [3-5] of sensitivity analysis starts with the algebraic system of equations (9). The differentiation of (9) with respect to b leads to the following system of equations

$$\frac{\mathrm{D}\mathbf{G}}{\mathrm{D}b}\mathbf{q} + \mathbf{G}\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}b} = \frac{\mathrm{D}\mathbf{H}}{\mathrm{D}b}\mathbf{T} + \mathbf{H}\frac{\mathrm{D}\mathbf{T}}{\mathrm{D}b} \tag{10}$$

or

$$\mathbf{G}\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}b} = \mathbf{H}\ \frac{\mathrm{D}\mathbf{T}}{\mathrm{D}b} + \frac{\mathrm{D}\mathbf{H}}{\mathrm{D}b}\mathbf{T} - \frac{\mathrm{D}\mathbf{G}}{\mathrm{D}b}\mathbf{q} \tag{11}$$

This approach of shape sensitivity analysis requires the differentiation of elements of matrices **G** and **H** with respect to the parameter b. The details concerning this problem are presented in [3].

It should be pointed out that in the system of equations (11) the values of **T** and **q** are known from the boundary conditions or basic problem solution (c.f. equation (9)). Differentiation of assumed boundary conditions (2) allows to calculate part of the values D**T**/Db, D**q** /Db, while the remaining part should be determined from (11).

4. Inverse problem and method of solution

The inverse problem considered here reduces to the assumption that one of the geometrical parameters of hole, this means radius of circle R or one of the circle center co-ordinate (x_s or y_s) is regarded as unknown, while the other quantities appearing in the equations (1)-(2) are assumed to be known.

Additionally, it is assumed, that the values of temperatures $T_{d\,i}$, $i = 1,2,...,M$ at the points (x_i, y_i) located on the surface Γ_1 resulting from adequate measurements are known. It should be pointed out that in this paper the temperatures $T_{d\,i}$ have been obtained from the direct problem (1), (2) solution for assumed values of x_s, y_s, R.

To solve the inverse problem formulated, the gradient method has been used [5, 6]. As previously, we assume that the parameter b corresponds to the radius R of internal hole or corresponds to the position of its centre, this means $b = x_s$ or $b = y_s$. The unknown geometrical parameter of internal hole can be estimated on the basis of minimization of the least squares criterion [7]

$$S = \sum_{i=1}^{M} (T_i - T_{d\,i})^2 \tag{12}$$

where T_i are the calculated temperatures at the points (x_i, y_i) located on the surface Γ_1. The calculated temperatures are obtained from the solution of direct problem given by equations (1), (2) by using the current available estimate for the parameter b. Using the necessary condition of optimum one obtains

$$\frac{\mathrm{D}\,S}{\mathrm{D}\,b} = 2\sum_{i=1}^{M} (T_i - T_{d\,i}) \frac{\mathrm{D}T_i}{\mathrm{D}\,b} = 0 \tag{13}$$

Function T_i is expanded in a Taylor series about known value of parameter b_k

$$T_i = T_i^k + \left.\frac{\mathrm{D}\,T_i}{\mathrm{D}\,b}\right|_{b=b_k} (b_{k+1} - b_k) \tag{14}$$

where T_i^k are the calculated temperatures under the assumption that $b = b_k$, where k is the number of iteration. It should be pointed out that b_0 is an arbitrary assumed value of parameter b, while for $k > 0$ it results from the previous iteration.

Putting (14) into (13) one has

$$\sum_{i=1}^{M}\left[T_i^{\,k}+\left.\frac{\mathrm{D}T_i}{\mathrm{D}b}\right|_{b=b_k}(b_{k+1}-b_k)-T_{di}\right]\left.\frac{\mathrm{D}T_i}{\mathrm{D}b}\right|_{b=b_k}=0 \tag{15}$$

or

$$(b_{k+1}-b_k)\sum_{i=1}^{M}\left[\left.\frac{\mathrm{D}T_i}{\mathrm{D}b}\right|_{b=b_k}\right]^2=\sum_{i=1}^{M}\left.\frac{\mathrm{D}T_i}{\mathrm{D}b}\right|_{b=b_k}(T_{di}-T_i^k) \tag{16}$$

this means

$$b_{k+1}=b_k+\frac{\displaystyle\sum_{i=1}^{M}\left.\frac{\mathrm{D}T_i}{\mathrm{D}b}\right|_{b=b_k}(T_{di}-T_i^k)}{\displaystyle\sum_{i=1}^{M}\left[\left.\frac{\mathrm{D}T_i}{\mathrm{D}b}\right|_{b=b_k}\right]^2} \tag{17}$$

The value b_{k+1} constitutes the input data for the next iteration. The iteration process is stopped when $k = K$, where K is the assumed number of iterations or when the predefined value of criterion (12) is achieved.

5. Results of computations

The square of dimensions 0.05 × 0.05 m is considered as shown in Figure 2. At first, the direct problem is solved under the assumption that the centre of the circle: $x_s = 0.025$, $y_s = 0.025$, radius: $R = 0.01$ m. It is assumed that $\lambda = 1$ W/ mK. On the bottom external boundary the Neumann condition $q_b = -10^4$ W/m^2 is accepted, on the remaining part of the external boundary the Dirichlet condition $T_{b1} = 500$°C is assumed. Along the circle the constant temperature $T_{b2} = 700$°C is given. The external boundary has been divided into 40 linear elements, while the internal boundary has been divided into 16 linear elements. In Figure 3 the temperature distribution at the bottom surface is shown.

Next, the inverse problems have been solved under the assumption that the temperatures at the boundary nodes 1,2, ..,11 (Fig. 2) are known (Fig. 3).

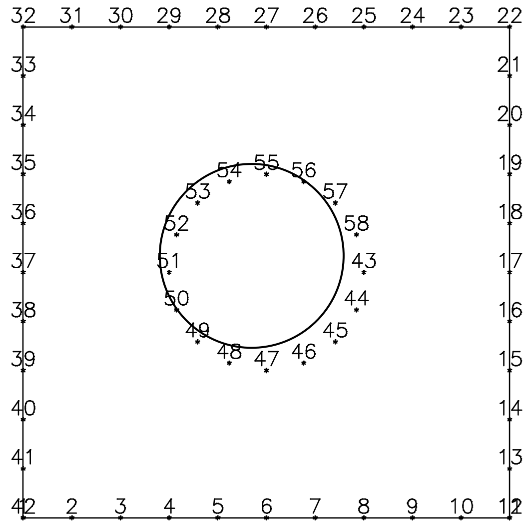

Fig. 2. Discretization

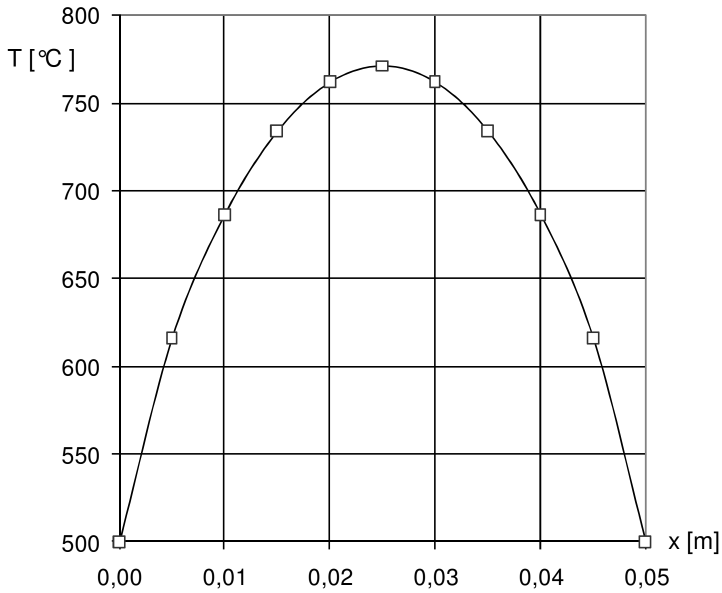

Fig. 3. Temperature distribution along the bottom surface

In Figure 4 the solutions for different initial values of identified parameters are shown. It is visible that the iteration processes described by equation (17) are convergent and the real values of parameters are obtained after a few iterations.

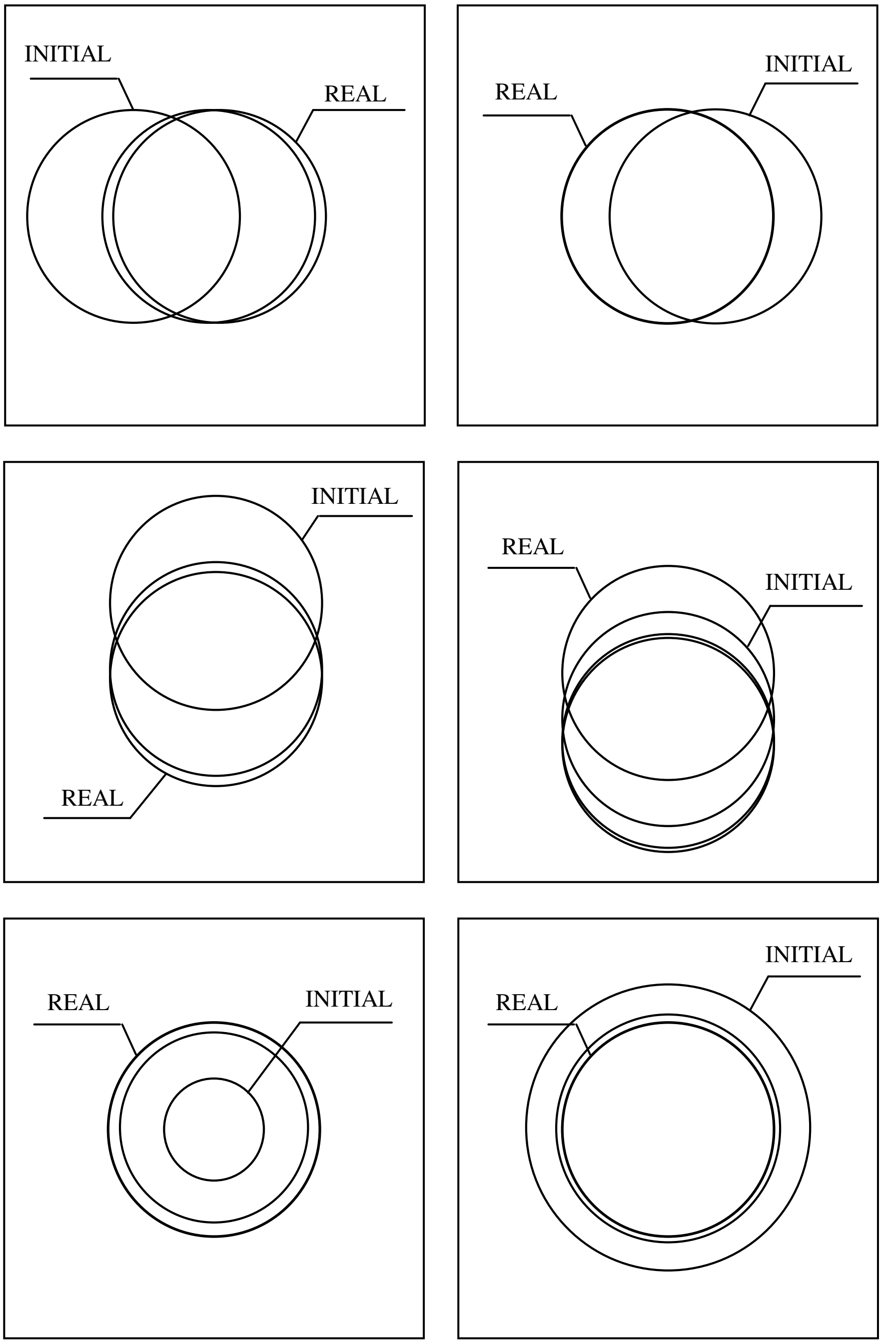

Fig. 4. Results of identification

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