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## IMAGE RECONSTRUCTION BY MEANS OF CHAOTIC ITERATIVE ALGORITHM

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**Abstract.** The image reconstruction problem from projection data for some particular schemes of reconstruction is considered. There are used the Algebraic Reconstruction Technique (ART) and the chaotic algebraic iterative algorithm CHART to reconstruct high-contrast objects from incomplete data. The influence of various parameters of these algorithms, such as the relaxation coefficient  $\lambda$ , the number of iterations, the number of projection data, and the noise in the projection data on reconstruction quality for different schemes of reconstruction is investigated. Numerical simulation results for a number of modeling high-contrast objects are presented and discussed.

### Introduction

Computed tomography allows to research the internal structure of an object without its destruction. This technique makes possible to reconstruct this structure using the projection data collected outside the object. In the recent years computed tomography found a wide application not only in medicine but also for solving many technical problems.

In all these applications we deal with inverse problems which are those of image reconstruction from projections. In spite of the wide different physical methods for collecting projection data there is a common mathematical nature of reconstruction problems: there is an unknown distribution of some physical parameter; a finite number of line integrals of this parameter can be estimated from physical measurements, and an estimate of the distribution of the original parameter is desired.

Theoretically it is possible to reconstruct the internal structure of an object uniquely without errors by means of the Radon inversion formula. If the number of projections is large enough and projections can be obtained for any orientation (in medicine, for example) then it is more preferable to use the analytical methods based on the inverse Radon transform or the Fourier transform [1-3]. However in practice projections are obtained by physical measurements and therefore they are given with some errors. Sometimes they cannot be available for each orientation and/or their number is very small, they may be sparse or missing at certain orientations (the limited angle problem). In all these cases one says that there is a problem of image reconstruction with incomplete projection data. In particular, such kind of

problems arise in mineral industries and engineering geophysics connected with acid drainage, stability of mine workers, mineral exploration and others [4, 5]. In these applications the algorithms based on analytical methods cannot be used [6]. The alternative reconstruction method which can be used under these conditions is the Algebraic Reconstruction Technique (ART), i.e. the iterative algebraic algorithms [7].

In this case the problem of image reconstruction is reduced to solving a system (possibly inconsistent) of linear algebraic equations, one for each projection ray, which has a few main characteristics: it is rectangular (not square as a rule), and it has a very large dimension. The problem of solving a large nonsingular system of algebraic equations is the most wide-spread and arises in many other applications. For solving such systems the different kinds of algebraic iterative algorithms are often used, the best known from which is the algorithm ART [7-10]. They are rather simple, flexible and permit to use *a priori* knowledge of the object before its reconstruction that is very important in many practical applications with incomplete projection data. Recently, these algorithms are also used in magnetic resonance imaging (MRI) [11, 12].

The outline of this paper is as follows. Section 1 considers the problem of image reconstruction from projections. For solving this problem the main algebraic iterative algorithm ART is used. In this section we also consider a chaotic iterative algebraic algorithm CHART, which is based on asynchronous methods and can be realized on a sequential computer. Section 2 is devoted to study the quality and convergence of the algebraic algorithms ART and CHART for image reconstruction of some modeling objects from incomplete projection data by means of computer simulation. Some numerical results of computer simulation which shows the comparing evaluations of errors and speed of convergence of these algorithms are presented in section 3. The influence of various parameters of these algorithms, such as the relaxation coefficient, the number of iterations, the number of projections, and the noise in the projections, on the reconstruction quality for different schemes of reconstruction are investigated. It is shown that the best results are obtained by means of the chaotic iterative algorithm CHART, which has the same robustness as the algorithm ART.

## 1. Algebraic algorithms of reconstruction

Let  $f(x,y)$  be a function which represents the spatial distribution of a physical parameter. If  $L: l = x \cos \theta + y \sin \theta$  is a line (ray) in the plane then the line integral

$$p_L = \int_L f(x, y) dL = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) \delta(l - x \cos \theta - y \sin \theta) dx dy \quad (1)$$

which is called a projection, is usually obtained from physical measurements.

In dependence on the obtaining system of projections there are many image reconstruction schemes, the main of them are parallel and beam schemes in the two-dimensional space.

From mathematical point of view the problem of reconstruction from projections is to find an unknown function  $f(x,y)$  by means of a given set of projections  $p_L$  for all  $L$ . This problem was solved by J. Radon in 1917 in the form of the inversion formula which expresses  $f$  in terms of  $p$ . Unfortunately, this mathematical problem represents only an idealized abstraction of problems which occur in practical applications. In practice only discrete projection data are given that estimate  $p$  for a finite number of rays. And it needs to find an image function  $f(x,y)$ , i.e. a reconstructed estimate of the unknown object. Moreover, we are deal with limited precision of the measured data and so the projection data are given with some errors. Therefore all these restrictions do not allow us to use the Radon inversion formula directly.

There exist two fundamentally different approaches for solving the image reconstruction problem. In the first approach the problem is formulated for continuous functions  $f$  and  $p$ , and the inversion formula is derived in this continuous model. This method is called the transform method approach [13]. The second approach is connected with discretization of the function  $f$  at the outset. Then the object  $f$  and measurements  $p$  become vectors in the finite dimensional Euclidean space. And for solving this model the methods of linear algebra and optimization theory are used. This approach is called the fully discretized model [13].

For constructing the fully discretized model a reconstructed domain is included into a rectangle  $E$  and it is divided into  $n$  small elements (pixels). The fully discretized model of the image reconstruction problem is based on the main principal that a distribution of a research object is equal to a constant value  $x_i$  throughout the  $i$ -th pixel. Denote by  $a_{ij}$  the length of the intersection of the  $i$ -th ray with  $j$ -th pixel. In this case the image reconstruction problem is reduced to solving a system of linear algebraic equations:

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{p} \quad (2)$$

where:

- $\mathbf{A} = (a_{ij}) \in \mathbf{R}^{m,n}$  is the matrix of coefficients,
- $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbf{R}^n$  is the image vector,
- $\mathbf{p} = (p_1, p_2, \dots, p_m)^T \in \mathbf{R}^m$  is the vector of projection data.

This system has a few characteristics: it is a rectangular, it has a very large dimension and  $\mathbf{A}$  is a sparse matrix. For solving this system it is often used different kinds of algebraic iterative algorithms the most well-known of which are the additive algorithm ART [1, 8-10]. These algorithms are very flexible and allow to apply different *a priori* information about object before its reconstruction that is especially very important when we have an incomplete projection data. Denote

$$\mathbf{P}_i(\mathbf{x}) = \mathbf{x} - \frac{(\mathbf{a}^i, \mathbf{x}) - p_i}{\|\mathbf{a}^i\|^2} \mathbf{a}^i \quad (3)$$

$$\mathbf{P}_i^\lambda = (1 - \lambda)\mathbf{I} + \lambda\mathbf{P}_i \quad (4)$$

where  $\mathbf{a}^i$  is the  $i$ -th row of the matrix  $\mathbf{A}$ , and  $\lambda$  is a relaxation parameter.

**Algorithm 1 (ART-1)**

1.  $\mathbf{x}^{(0)} \in \mathbf{R}^n$  is an arbitrary vector;
2. The  $k+1$ -th iteration is calculated in accordance with the following scheme:

$$\mathbf{x}^{(k+1)} = \mathbf{C}\mathbf{P}_i^{\lambda_k} \mathbf{x}^{(k)} \quad (i = 1, 2, \dots, m) \quad (5)$$

where  $\mathbf{P}_i^{\lambda_k}$  are operators defined by (3), (4)  $\lambda_k$  are relaxation parameters,  $\mathbf{C}$  is a constraining operator, and  $i(k) = k(\bmod m) + 1$ .

Algorithm 1 was proposed by Kaczmarz [14] and independently discovered and investigated by Herman, Lent, Rowland in [8]. It was used successfully in application of computerized tomography in medicine. This algorithm runs through all equations cyclically with modification of the present estimate  $\mathbf{x}^{(k)}$  in such a way that the present equation with index  $i$  is fulfilled.

Applying the general model of asynchronous iterations (see [15, 16]) for implementation of the algebraic iterative algorithm ART-1 there results an asynchronous iterative algorithm which particular case has the following form (see [17]):

**Algorithm 1 (CHART-1)**

1.  $\mathbf{x}^{(0)} \in \mathbf{R}^n$  is an arbitrary vector;
2. The  $k+1$ -th iteration is calculated in accordance with the following scheme:

$$\mathbf{y}^{k+1,i} = \begin{cases} \mathbf{P}_i^{\lambda_k} \mathbf{x}^{(k)}, & \text{if } i \in I_k \\ \mathbf{y}^{k,i}, & \text{otherwise} \end{cases}$$

$$\mathbf{x}^{(k+1)} = \mathbf{C}\mathbf{y}^{k+1,i} \quad (i = 1, 2, \dots, m) \quad (6)$$

where  $\mathbf{P}_i^{\lambda_k}$  are operators defined by (3), (4)  $\lambda_k$  are relaxation parameters,  $I = \{I_k\}_{k=1}^\infty$  is a sequence of chaotic sets such that  $I_k \subset \{1, 2, \dots, m\}$ ,  $\mathbf{C}$  is a constraining operator.

In this algorithm the numbers of equations in system (2) are chosen in a chaotic way.

## 2. Computer simulation

In order to evaluate the goodness of the computer reconstruction of high-contrast objects from incomplete data different kinds of geometric figures and reconstruction schemes were tested. For obtaining a good reconstruction in all experimental results it is very important, of course, the choice of main parameters, such as the number of pixels and the number of projection data, the use of a priori information, which can be defined by means of constraining operator  $C$ , the choice of the relaxation coefficient  $\lambda$ .

**Reconstruction schemes.** In some practical problems, for example in engineering, it is impossible to obtain projections from all orientations because of the existing some important reasons (such as situation, size or impossibility of an access to a research object). This situation arises, for example, in the coal bed working. In these cases the access to longwalls may be very difficult or impossible at all. Sometimes it is impossible to access to one or two sides of longwalls, and sometimes it is impossible to access only to the basis but all the longwalls are accessible. Each situation has own scheme of obtaining information.

In this paper we present results of image reconstructions only for two different reconstruction schemes for obtaining projection data, which is shown in Figure 1. In the first case there is an access to a research object from only two opposite sides. Therefore the sources of rays can be situated only on one side and the detectors are situated on the opposite side of the research part of a coal bed. This scheme of obtaining information will be called the system  $(1 \times 1)$ . And in the second case there is an access to all four sides of an object. Therefore the sources can be situated, for example, onto two neighboring sides, and the detectors can be situated on the opposite sides. So the projections can be obtained from two pair of the opposite sides. This situation will be called the system  $(1 \times 1, 1 \times 1)$ .

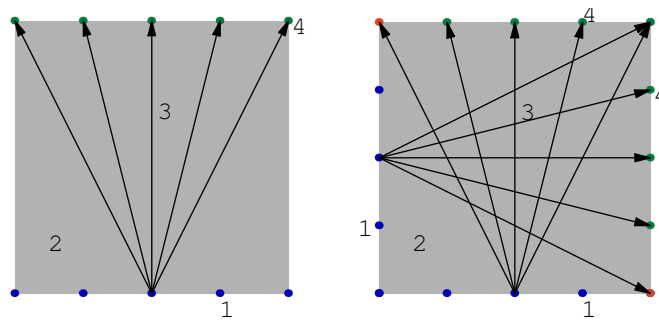


Fig. 1. Schemes for obtaining projections data - the system  $(1 \times 1)$  (on the left side) and the system  $(1 \times 1, 1 \times 1)$  (on the right side): 1 - sources of rays; 2 - a research object; 3 - rays; 4 - detectors

**Modeling functions.** An important factor in the simulation process of image reconstruction is the choice of modeling objects which describe the density distribution of research objects. In a coal bed, where we search the reservoirs of compressed gas or interlayers of a barren rock, the density distribution may be considered discrete and the density difference of these three environments (coal, compressed gas and barren rock) is significant. Therefore the discrete functions with high contrast were chosen for illustration of the implementation of the algorithms working with incomplete data. In this paper we present only the results of computer reconstruction for two discrete functions.

The first discrete function  $f_1(x, y)$  is given in the following form:

$$f_1(x, y) = \begin{cases} 1, & (x, y) \in D \subset E \subset \mathbf{R}^2, \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

where  $E$  is a square  $E = \{(x, y) : -1 \leq x, y \leq 1\}$ , and  $D$  is a subset of  $E$  of the following form:

$$D = [-0.4, -0.2] \times [-0.5, 0.5] \cup [-0.2, 0.2] \times [0.3, 0.5] \cup [-0.2, 0.2] \times [-0.1, 0.1] \cup [0, 0.2] \times [0.1, 0.3].$$

The second discrete function  $f_2(x, y)$  is given in the following form:

$$f_2(x, y) = \begin{cases} 1, & (x, y) \in D_1 \subset E \subset \mathbf{R}^2, \\ 2, & (x, y) \in D_2 \subset E \subset \mathbf{R}^2, \\ 3, & (x, y) \in D_3 \subset E \subset \mathbf{R}^2, \\ 4, & (x, y) \in D_4 \subset E \subset \mathbf{R}^2, \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

where  $E$  is a square  $E = \{(x, y) : -1 \leq x, y \leq 1\}$ , and  $D_i$  are subsets of  $E$  of the following form:

$$D_1 = [-0.7, -0.4] \times [-0.5, 0.2], \quad D_2 = [-0.2, 0.2] \times [-0.1, 0.1],$$

$$D_3 = [-0.2, 0.2] \times [0.3, 0.5], \quad D_4 = [0.4, 0.7] \times [0.4, 0.7].$$

The three-dimensional view of the plots of these functions are given in Figure 2.

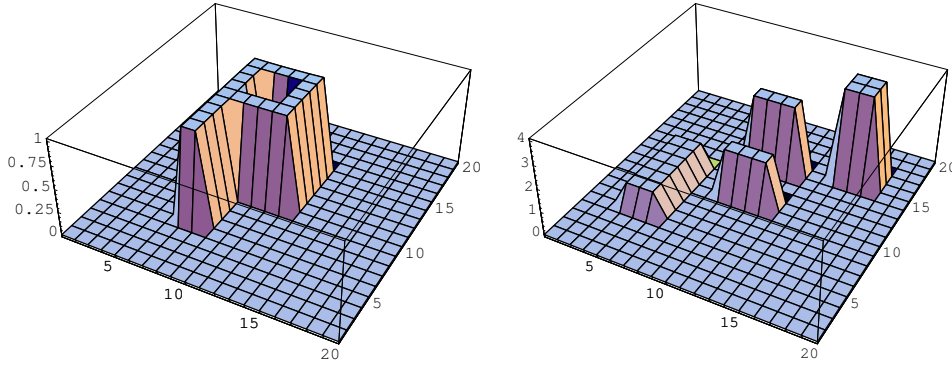


Fig. 2. The original functions  $f_1(x, y)$  (on the left side) and  $f_2(x, y)$  (on the right side)

**Constraining operator.** In all considered computer simulations it was assumed that  $\mathbf{C} = C_1 C_2$ , where

$$(C_1[\mathbf{x}])_i = \begin{cases} a, & \text{if } x_i < a; \\ x_i, & \text{if } a \leq x_i \leq b; \\ b, & \text{if } x_i > b; \end{cases} \quad (9)$$

$$(C_2[\mathbf{x}])_j = \begin{cases} 0, & \text{if } p_i = 0 \text{ and } a_{ij} \neq 0; \\ x_j, & \text{otherwise} \end{cases} \quad (10)$$

### Implementation of reconstruction algorithms

In the reconstruction algorithms ART-1 and CHART-1 it is very important the choice of the matrix  $\mathbf{A}$ . In our experiments the element  $a_{ij}$  is equal to the length of the intersection of the  $i$ -th pixel with  $j$ -th ray.

For the algorithm CHART-1 the sequence of chaotic sets  $I_k$  has the form  $\{\xi_k\}$ , where  $\xi_k$  is an integer random variable in the interval  $[1, m]$  with uniform distribution.

All algorithms were implemented on IBM/PC (processor AMD Duron XP, 1600 MHz) by means of C++ and MATHEMATICA 5.1.

### 3. Numerical results

In this section we present the numerical results of image reconstruction of the functions given by (7) and (8) with the algebraic iterative algorithms ART-1 and CHART-1 for the reconstruction schemes  $(1 \times 1)$  and  $(1 \times 1, 1 \times 1)$ . We compare the reconstruction results and investigate the influence of various parameters of these algorithms such as a pixel initialization, the relaxation coefficient, the number of iterations and noise in the projection data on reconstruction quality. The conver-

gence of algorithms ART-1 and CHART-1 was studied in dependence on all these parameters.

The choice of the relaxation coefficient  $\lambda$  in the iterative algebraic algorithms of reconstruction is one of the most important. Its choice depends on the other parameters, such as the number of pixels, the number of rays and the system of reconstruction. In this paper for all algorithms we consider that  $\lambda_k = \lambda = \text{const}$  for all  $k$ . From the conducted research it follows that the optimal value of a relaxation parameter  $\lambda$  is equal to 1.1 for the system  $(1 \times 1, 1 \times 1)$  and it is equal to 1.3 for the system  $(1 \times 1)$  in the case without noise in projections. In the case when there is a noise in the projection data the optimal value of  $\lambda$  is changed in dependence on the value of the noise.

In all presented numerical results it was assumed, that

- $n$  - is the number of pixels, i.e. the number of variables,
- $m$  - is the number of rays, i.e. the number of equations,
- $iter$  - is the number of full iterations,
- the sequence of chaotic sets  $I_k$  has the form  $\{\xi_k\}$ , where  $\xi_k$  is an integer random variable in the interval  $[1, m]$  with uniform distribution,
- the reconstruction domain  $E = \{(x, y) : -1 \leq x, y \leq 1\}$  was divided into  $n = 20 \times 20$  pixels,
- the number of projections  $m$  in the system  $(1 \times 1)$  is equal to 788, and in the system  $(1 \times 1, 1 \times 1)$  the number  $m = 644$ .

In this paper the convergence characteristics of image reconstruction are presented in a view of plots for the following measures of errors:

- the maximal absolute error

$$\Delta = \max_i |f_i - \tilde{f}|$$

- the error

$$\delta(x) = |f(x) - \tilde{f}(x)|$$

- the mean absolute error

$$\delta_2 = \frac{1}{n} \sum_i |f_i - \tilde{f}_i|$$

where  $f_i$  is the value of a given modeling function in the center of the  $i$ -th pixel and  $\tilde{f}_i$  is the value of the reconstructed function in the  $i$ -th pixel.

The reconstruction result of the function  $f_1(x, y)$  with algorithm CHART-1 after 100 iterations in the system  $(1 \times 1, 1 \times 1)$  is shown in Figure 3.



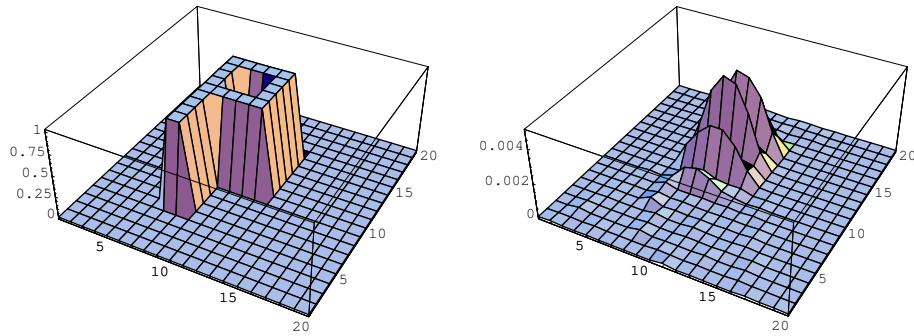


Fig. 3. Reconstruction and the error  $\delta(x)$  of  $f_1(x, y)$  obtained with chaotic algorithm CHART-1 for  $n = 20 \times 20$ ,  $m = 788$ ,  $iter = 100$  in the system  $(1 \times 1)$

The plot, which is presented in Figure 4, illustrates the dependence of the mean absolute error  $\delta_2$  on the number of iterations for image reconstruction of  $f_1(x, y)$  with algorithms ART-1 and CHART-1 in the systems  $(1 \times 1, 1 \times 1)$  and  $(1 \times 1)$ .

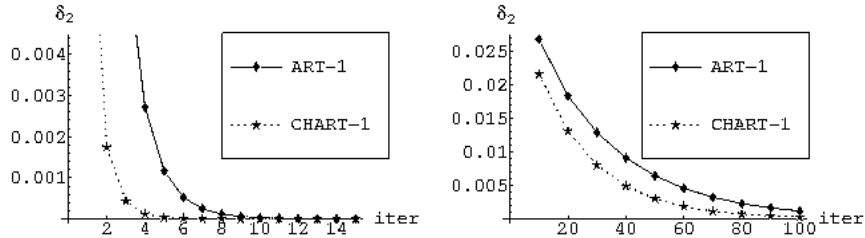


Fig. 4. Dependence of the mean absolute error  $\delta_2$  on the number of iterations for image reconstruction of  $f_1(x, y)$  with algorithms ART-1 and CHART-1 in the system  $(1 \times 1, 1 \times 1)$  (on the left side) and in the system  $(1 \times 1)$  (on the right side)

The result of image reconstruction of the function  $f_2(x, y)$  with chaotic algorithm CHART-1 after 25 iterations in the system  $(1 \times 1)$  is shown in Figure 5.

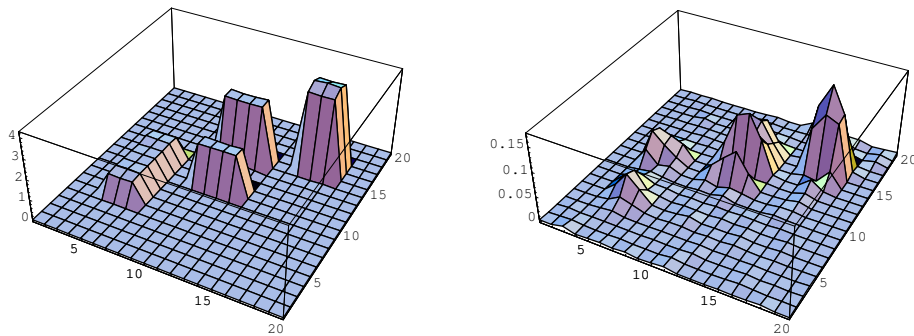


Fig. 5. Reconstruction and the error  $\delta(x)$  of  $f_2(x, y)$  obtained with chaotic algorithm CHART-1 for  $n = 20 \times 20$ ,  $m = 788$ ,  $iter = 25$  in the system  $(1 \times 1)$

Table 1 (resp. Table 2) shows the dependence of the maximum absolute error  $\Delta$  on the number of iterations for the algorithms ART-1 and CHART-1 for the system  $(1 \times 1)$  (resp., the system  $(1 \times 1, 1 \times 1)$ ) and the same set of parameters which were chosen above.

Table 1. The dependence of the maximum absolute error  $\Delta$  on the number of iterations for algorithms ART-1 and CHART-1 in the system  $(1 \times 1)$

<i>iter</i>	ART-1	CHART-1
100	0.0306	0.0073
200	0.00201	0.0001
500	$1.209 \times 10^{-6}$	$4.098 \times 10^{-9}$
10000	$6.435 \times 10^{-12}$	$6.328 \times 10^{-15}$

Table 2. The dependence of the maximum absolute error  $\Delta$  on the number of iterations for algorithms ART-1 and CHART-1 in the system  $(1 \times 1, 1 \times 1)$

<i>iter</i>	ART-1	CHART-1
10	0.0077	0.00002
20	$9.83 \times 10^{-6}$	$3.568 \times 10^{-9}$
40	$3.12 \times 10^{-11}$	$1.221 \times 10^{-15}$
50	$3.98 \times 10^{-14}$	$1.11 \times 10^{-15}$
100	$8.88 \times 10^{-16}$	$8.88 \times 10^{-16}$

It is very important for the research of reconstruction quality to consider the effect of noise in projection data on a reconstructed image. For this purpose the various levels of Gaussian (normal) noise are added to projections. The noise model was proposed by Herman [9]. In this model the noisy data are simulated by the addition of Gaussian noise. Here each projection is multiplied by a Gaussian distributed random number with mean 1.0 and standard deviation  $s$ . Usually the noise level is taken less than 10% in computed tomography for non-medical imaging application. In considered experiments the noise level was in the interval from 1% to 5%.

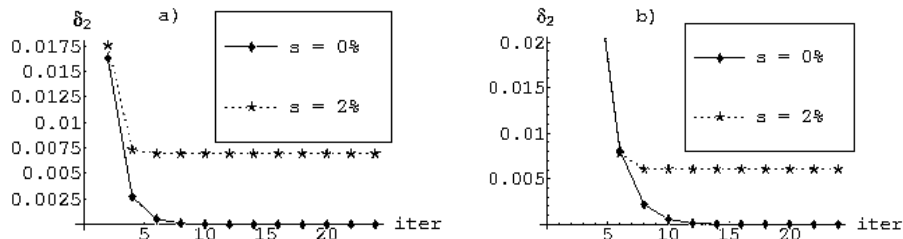


Fig. 6. Dependence of the mean absolute error  $\delta_2$  on the number of iterations for image reconstruction of  $f_1(x, y)$  with algorithm ART-1 (on the left side) and CHART-1 (on the right side) in the system  $(1 \times 1, 1 \times 1)$  and the normal noise of projection data  $s = 0\%$  and  $s = 2\%$

The plots, which are represented in Figure 6, show dependence of the mean absolute error  $\delta_2$  on the number of iterations for image reconstruction of  $f_1(x, y)$  with algorithm ART-1 (on the left side) and algorithm CHART-1 (on the right side) in the system  $(1 \times 1, 1 \times 1)$  and the normal noise in projections  $s = 0\%$  and  $s = 2\%$ . These results shows that both algorithms are stable with regard to noise in projections and they have the same robustness.

## Conclusions

The aim of this paper was to represent and study the iterative algebraic algorithms for reconstruction of high-contrast objects. There were studied the quality and convergence of these algorithms by computing simulation on sequential computer.

The obtained results show also that for obtaining the good reconstruction quality (with  $\delta_1 < 1\%$  and  $\delta_2 < 0.001$ ) by means of the chaotic algorithm CHART-1 it suffices 4-6 iterations in the system  $(1 \times 1, 1 \times 1)$  and 20-30 iterations in the system  $(1 \times 1)$ . These results are much better by comparison with the algorithm ART-1.

All experimental results show that the errors of reconstruction of considered objects from limited projection data with all considered algorithms are constantly reduced with increasing the number of iterations. It was also shown that all considered algorithms are stable with regard to noise ( $< 5\%$ ) in projections.

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