PSEUDOGROUPS OF TRANSFORMATIONS ON ANTIDISCRETE TOPOLOGICAL SPACES

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Abstract. In this paper we show that groups of transformations defined on antidiscrete topological spaces are the only ones among pseudogroups. It means that if a topological structure of a pseudogroup is the weakest then an algebraic structure of this pseudogroup is a group.

Introduction

The notion of a pseudogroup was formed progressively together with the development of differential geometry. First mathematicians who realized a classic notion of a group of transformations was not sufficient for purposes of differential geometry were O. Veblen and J.H.C. Whitehead in 1932. Their definition was improved by J.A. Schouten and J.Haantjes in 1937, S. Gołąb in 1939 and C. Ehresmann in 1947. Gołąb's and Ehresmann's definitions are good enough to be used at present. It was shown in [1] that axioms of Ehresmann's definition can be formulated in the equivalent way which simplyfy proofs. We used this definition in [2] to show that a group of transformations can be treated as a pseudogroup.

Main result

Let us remind the following version of Ehresmann's definition which is used in differential geometry and can be found in [3].

Definition 1. A pseudogroup of transformations on a topological space S is a set Γ of transformations satisfying the following axioms:

1° Each $f \in \Gamma$ is a homeomorphism of an open set of S onto another open set of S;

2° If $f \in \Gamma$, then the restriction of f to an arbitrary open subset of the domain of f is in Γ;

3° Let $U = \bigcup_i U_i$ where each U_i is an open set of. S. A homeomorphism f of U onto an open set of S belongs to Γ if the restriction of f to U_i is in Γ for every i;

4° For every open set U of S, the identity transformation of U is in Γ;

5° If $f \in \Gamma$, then $f^{-1} \in \Gamma$;

6° If $f \in \Gamma$ is a homeomorphism of U onto V and $g \in \Gamma$ is a homeomorphism of Y onto Z and if $V \cap Y$ is non-empty, then the homeomorphism $g \circ f$ of $f^{-1}(V \cap Y)$ onto $g(V \cap Y)$ is in Γ.

We will also use the following definition introduced in [1].

Definition 2. A non-empty set Γ of functions for which domains D_f are arbitrary non-empty sets will be called a pseudogrup of functions if it satisfies the following conditions:

1° $f(D_f) \cap D_g \neq \varnothing \Rightarrow g \circ f \in \Gamma$ for $f,g \in \Gamma$

2° $f^{-1} \in \Gamma$ for $f \in \Gamma$

3° $\bigcup \Gamma' \in \Gamma$ for $\Gamma' \in \langle\Gamma\rangle$

where

$$\langle\Gamma\rangle = \left\{\varnothing \neq \Gamma' \subset \Gamma : \bigcup\Gamma' \text{ is a function and } \bigcup(\Gamma')^{-1} \text{ is a function}\right\}$$

and

$$(\Gamma')^{-1} = \left\{f^{-1} : f \in \Gamma'\right\}$$

and f^{-1} denotes an inverse relation.

It was shown in [1] that if Γ is a pseudogrup, then $(\bigcup_{f\in\Gamma} D_f, \{D_f : f \in \Gamma\} \cup \{\varnothing\})$ is a topological space and Γ is an Ehresmann pseudogrup of transformations on this topological space. On the other hand, if Γ is an Ehresmann pseudogrup of transformations on a topological space S, then Γ is a pseudogrup of functions.

We will need the following definition which was introduced in [4].

Definition 3. A generalized inverse semigroup is a partial groupoid $(B,\cdot)$ satisfying the following axioms:

1° $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

holds when one of the sides is defined;

$2°$ For every $a \in B$ there exists exactly one $b \in B$ such that

$$a \cdot (b \cdot a) = a \qquad \text{and} \qquad b \cdot (a \cdot b) = b$$

It is not difficult to see that thanks to definition 2 we get a connection between groups of transformations and pseudogroups. It is also easy to see that groups of transformations are the only pseudogroups on antidiscrete topological spaces. We will use definition 1 to prove it.

Theorem. If Γ is a pseudogroup of transformations on a topological space S which is antidiscrete then Γ is a group of transformations.

Proof. From $1°$ it follows that Γ is non-empty. Because the topological space is antidiscrete there is only one non-empty open set i.e. the whole space. From $2°$ it follows that all functions are transformations of an open set onto the same open set. From $5°$ it follows that the inverse transformation belongs to Γ for every $f \in \Gamma$. From $6°$ it follows that the composition of two transformations belonging to Γ belongs to Γ. That requires.

Conclusions

Of course we can replace a pseudogroup of transformations by a pseudogroup of functions and the theorem will be true. It was shown in [5] that even a Schouten-Haantjes pseudogroup is a generalized inverse semigroup. As the definition of Schouten-Haantjes is more general we can also say that a pseudogroup of functions is a generalized inverse semigroup. The above theorem shows that in the case of pseudogroups on antidiscrete topological spaces the topological structure influences on the algebraic structure because we obtain something more than a generalized inverse semigroup. A pseudogroup is a group when topolgical structure is the weakest. It was shown in [3] that we have the inverse theorem i.e. when a pseudogroup is a group then a topological structure is the weakest.

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